Last Time

Wrote down the Grn-fcn of the wave eqn

$$-\Box u(t,\vec{r}) = J(t,\vec{r})$$

$$-\left(\frac{-1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right) u(t,\vec{r}) = J(t,\vec{r})$$

Then the Grn-fcn satisfies

$$-\left(\frac{-1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right) G(t\vec{r}\,|\,t_0\vec{r}_0) = \delta(t-t_0)\,\delta^3(\vec{r}-\vec{r}_0)$$

And we found it

$$\boxed{G(t,\vec{r}\,|\,t_0\vec{r}_0) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|}\,\delta\left(t-t_0-\frac{|\vec{r}-\vec{r}_0|}{c}\right)}$$

So

$$u(t,\vec{r}) = \int d^3r_0\,dt_0\,\frac{1}{4\pi|\vec{r}-\vec{r}_0|}\,\delta\left(t-t_0-\frac{|\vec{r}-\vec{r}_0|}{c}\right) J(t_0,\vec{r}_0)$$

$$\boxed{u(t,\vec{r}) = \int d^3r_0\,\frac{1}{4\pi|\vec{r}-\vec{r}_0|}\,J\left(t-\frac{|\vec{r}-\vec{r}_0|}{c},\,\vec{r}_0\right)}$$

kind of Coulomb law but source is evaluated at the retarded time $T = t - \frac{|\vec{r}-\vec{r}_0|}{c}$

Last Time (Continued)

The methodolog was also important.

- We talked about retarded Grn-fcns more generally

$$\left[m \frac{d^2}{dt^2} + m\omega_0^2 \right] G_R(t, t_0) = \delta(t - t_0)$$

- Then showed (in two ways) that

$$G_R(t) = \Theta(t) \frac{\sin \omega_0 t}{m\omega_0}$$

$$G_R(\omega) = \frac{1}{m} \frac{1}{\left[-(\omega + i\varepsilon)^2 + \omega_0^2 \right]}$$

- For the wave eqn $G_R(t, k) \leftarrow G_R$ in time and k

$$\frac{1}{c^2} \left[\frac{\partial^2}{\partial t^2} + (ck)^2 \right] G_R(t, k) = \delta(t - t_0)$$

Thus the wave-eqn is a SHO for each fourier mode

$$G_R(t, k) = c^2 \Theta(t) \frac{\sin(ckt)}{ck}$$

$$G_R(\omega, k) = \frac{c^2}{\left[-(\omega + i\varepsilon)^2 + (ck)^2 \right]}$$

Maxwell Eqs + Waves

$$\nabla \cdot E = \rho$$

$$\nabla \times B = \frac{\vec{J}}{c} + \frac{1}{c}\partial_t \vec{E}$$

$$\nabla \cdot B = 0$$

$$-\nabla \times E = \frac{1}{c}\partial_t \vec{B}$$

Introduce φ and $\vec{A}$, $B = \nabla \times \vec{A}$, $\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla\varphi$
and found

$$-\Box\varphi - \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\varphi + \nabla\cdot\vec{A}\right) = \rho$$

$$-\Box\vec{A} + \vec{\nabla}\left(\frac{1}{c}\partial_t\varphi + \nabla\cdot A\right) = \vec{J}/c$$

Selecting the Lorentz Gauge:

$$\frac{1}{c}\partial_t\varphi + \nabla\cdot\vec{A} = 0$$

Find

wave eqns

$$-\Box\varphi = \rho$$

$$-\Box\vec{A} = \vec{J}/c$$

Then the Grn-fcns solutions give

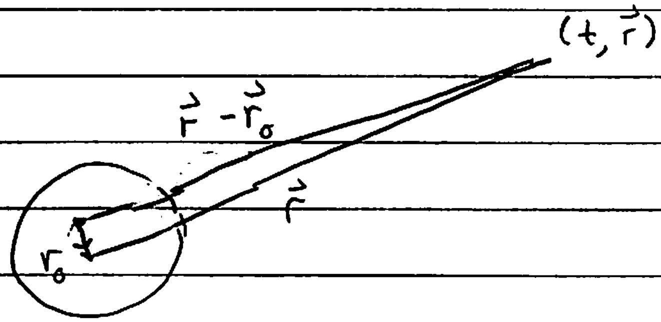
$$\vec{A}(t,\vec{r}) = \int \frac{d^3 r_0}{4\pi |\vec{r}-\vec{r}_0|} \frac{\vec{j}}{c}\left(t - \frac{|\vec{r}-\vec{r}_0|}{c}, \vec{r}_0\right)$$

$$\varphi(t,\vec{r}) = \int \frac{d^3 r_0}{4\pi |\vec{r}-\vec{r}_0|} \rho\Big(\underbrace{t - \frac{|\vec{r}-\vec{r}_0|}{c}}_{\text{retarded time}}, \vec{r}_0\Big)$$

Long Distance Expansion

- At great distances

$$|\vec{r} - \vec{r}_0| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}_0 + r_0^2} \simeq r - \hat{n}\cdot r_0 \qquad \vec{n} = \hat{r}$$



So

$$T = t - \frac{|\vec{r} - \vec{r}_0|}{c} \simeq t - \frac{r}{c} + \frac{n\cdot\vec{r}_0}{c}$$

Valid expression whenever in far zone

So find in the far field up to $1/r^2$

$$\vec{A}(t,\vec{r}) = \frac{1}{4\pi r}\int d^3r_0 \, \frac{\vec{J}}{c}(T, r_0)$$

$$\varphi(t,r) = \frac{1}{4\pi r}\int d^3r_0 \, \rho(T, r_0)$$

$$\frac{1}{4\pi|r - r_0|} \simeq \frac{1}{4\pi r}$$

So then we need, $B = \nabla \times \vec{A}$ and $E = -\frac{1}{c}\partial_t \vec{A} - \nabla\varphi$

Computing the Fields (pg. 1)

- To compute the fields we need to differentiate w.r.t. $t, \vec{r}$. Note that

$$T(t,r) = t - \frac{r}{c} + \frac{\vec{n}\cdot r_0}{c} \quad \text{is a function of } t, r$$

$$\underline{\underline{B = \nabla \times A}}$$

- Note, $\frac{\partial}{\partial r^i}\frac{1}{r} = -\frac{1}{r^2} n_i$ ← suppressed by extra $\frac{1}{r}$ compared to $\frac{1}{r}$

- $$\left(\frac{\partial J(T, r_0)}{\partial r^i}\right)_t = \left(\frac{\partial J}{\partial T}\right)_{r_0} \frac{\partial T}{\partial r^i} = -\frac{\partial \vec{J}}{\partial T}\frac{n_i}{c}$$

$$T = t - \frac{r}{c} + \frac{\vec{n}\cdot r_0}{c}$$

$$\frac{\partial T}{\partial r^i} = \frac{-n_i}{c} + \left(\frac{\partial}{\partial r^i}\vec{n}\right)\cdot\frac{\vec{r}_0}{c}$$

$$\simeq -\frac{n_i}{c}$$

$O\left(\frac{1}{r}\right)$ so can discard

So $B^i = \varepsilon^{ijk}\frac{\partial A_k}{\partial r^j}$ gives

$$\boxed{\vec{B} = \frac{-\vec{n}/c}{4\pi r} \times \int_{r_0} \frac{1}{c}\left(\frac{\partial \vec{J}(T, r_0)}{\partial T}\right)_{r_0}}$$

Computing The Fields pg. 2

$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla_r \varphi$

This works the same way $\quad \partial T/\partial r^i = -\frac{n_i}{c}$

$$-\nabla_r \rho(T, r_0) = -\frac{\partial \rho}{\partial T}(T, r_0) \overbrace{\nabla_r T}$$

$$= \frac{\partial \rho}{\partial T}(T, r_0) \frac{\vec{n}}{c}$$

So

$$\vec{E} = -\frac{1}{c}\frac{1}{4\pi r}\int_{r_0} \underbrace{\frac{1}{c}\frac{\partial \vec{J}}{\partial T}}_{\text{from } \partial_t \vec{A}} + \underbrace{\frac{\vec{n}}{c}\frac{\partial \rho}{\partial T}(T, r_0)}_{\text{and } \nabla_r \varphi}$$

Using current conservation, (see techical note)

$$\left(\frac{\partial \rho}{\partial T}\right)_{r_0} = -\left(\nabla \cdot \vec{J}\right)_{r_0} = -\left(\frac{\partial \vec{J}}{\partial \vec{r}_0}\right)_t + \frac{\vec{n}}{c}\cdot\left(\frac{\partial \vec{J}}{\partial t}\right)_{r_0}$$

Fixed T $\qquad$ fixed time

Find

$$\vec{E} = -\frac{1}{c}\frac{1}{4\pi r}\int_{r_0} \frac{1}{c}\frac{\partial \vec{J}}{\partial t}(T, r_0) - \frac{\vec{n}}{c}\, \vec{n}\cdot\frac{\partial \vec{J}}{\partial t}(T, r_0)$$

$$= \frac{-1}{4\pi r\, c}\int d^3 r_0 \, \underbrace{\left(\partial_t \vec{J} - \vec{n}\, \vec{n}\cdot\partial_t \vec{J}\right)/c}_{\text{transverse piece of } \vec{j}}$$

or

$$\vec{E} = \frac{1}{4\pi r c} \vec{n} \times \vec{n} \times \int d^3 r_0 \, \partial_t \vec{J} / c$$

$$\vec{E} = -\vec{n} \times \vec{B}$$

Summary:

① We solved for potentials and fields in the far field

$$A_{rad} = \frac{1}{4\pi r} \int d^3 r_0 \, \vec{J}(T, r_0) / c$$

$$\vec{B} = \frac{-\vec{n} \times}{4\pi r c} \int d^3 r_0 \, \partial_t \vec{J}(T, r_0) / c$$

$$\vec{E} = \frac{\vec{n} \times \vec{n} \times}{4\pi r c} \int d^3 r_0 \, \partial_t J(T, r_0) / c$$

$\vec{n} \times \vec{n} \times \partial_t \vec{J}$ = transvese piece of $\partial_t J$

Computing The Fields (Technical Aside) (pg.3)

We have changed variables from t, r_0 to T, r_0

$t, r_0 \longrightarrow T, r_0$

$$T = t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{r}_0}{c} \quad \Longleftrightarrow \quad t = T + \frac{r}{c} - \frac{n\cdot\vec{r}_0}{c}$$

$$\vec{r}_0 = \vec{r}_0$$

So

$$\left(\frac{\partial}{\partial T}\right)_{r_0} = \left(\frac{\partial}{\partial t}\right)_{r_0} \overbrace{\left(\frac{\partial t}{\partial T}\right)_{r_0}}^{1} + \left(\frac{\partial}{\partial \vec{r}_0}\right)_t \cdot \cancelto{0}{\left(\frac{\partial \vec{r}_0}{\partial T}\right)_{r_0}}$$

$$\left(\frac{\partial}{\partial T}\right)_{r_0} = \left(\frac{\partial}{\partial t}\right)_{r_0} \Longrightarrow \boxed{\frac{\partial}{\partial T} = \frac{\partial}{\partial t}}$$

and

$$\left(\frac{\partial}{\partial \vec{r}_0}\right)_T = \left(\frac{\partial}{\partial \vec{r}_0}\right)_t \frac{\partial \vec{r}_0}{\partial \vec{r}_0} + \left(\frac{\partial}{\partial t}\right)_{r_0} \left(\frac{\partial t}{\partial \vec{r}_0}\right)_T$$

$$\left(\frac{\partial}{\partial r_0}\right)_T = \left(\frac{\partial}{\partial r_0}\right)_t - \frac{n}{c}\cdot\left(\frac{\partial}{\partial t}\right)_{r_0}$$

$$\boxed{\left(\frac{\partial}{\partial \vec{r}_0}\right)_T = \left(\frac{\partial}{\partial \vec{r}_0}\right)_t - \frac{n}{c}\cdot\left(\frac{\partial}{\partial t}\right)_{\vec{r}_0}}$$