11 Radiation in Non-relativistic Systems

11.1 Basic equations

This first section will NOT make a non-relativistic approximation, but will examine the far field limit.

(a) We wrote down the wave equations in the covariant gauge:

\[-\Box \Phi = \rho(t_0, r_0)\]  (11.1)
\[-\Box A = J(t_0, r_0)/c\]  (11.2)

The gauge condition reads

\[\frac{1}{c} \partial_t \Phi + \nabla \cdot A = 0\]  (11.3)

(b) Then we used the green function of the wave equation

\[G(t, r|t_0, r_0) = \frac{1}{4\pi |r - r_0|} \delta(t - t_0 + \frac{|r - r_0|}{c})\]  (11.4)

to determine the potentials \((\Phi, A)\)

\[\Phi(t, r) = \int d^3 x_o \frac{1}{4\pi |r - r_o|} \rho(T, r_o)\]  (11.5)
\[A(t, r) = \int d^3 x_o \frac{1}{4\pi |r - r_o|} J(T, r_o)/c\]  (11.6)

Here \(T(t, r)\) is the retarded time

\[T(t, r) = t - \frac{|r - r_o|}{c}\]  (11.7)

(c) We used the potentials to determine the electric and magnetic fields. Electric and magnetic fields in the far field are

\[A_{\text{rad}}(t, r) = \frac{1}{4\pi r} \int_{r_o} J(T, r_o)/c\]  (11.8)

and

\[B(t, r) = -\frac{n}{c} \times \partial_t A_{\text{rad}}\]  (11.9)
\[E(t, r) = n \times \frac{n}{c} \times \partial_t A_{\text{rad}} = -n \times B(t, r)\]  (11.10)

In the far field (large distance limit \(r \to \infty\)) limit we have

\[T = t - \frac{r}{c} + \frac{n \cdot r_o}{c}\]  (11.11)

And we recording the derivatives

\[\left( \frac{\partial}{\partial t}\right)_{r_o} = \left( \frac{\partial}{\partial T}\right)_{r_o}\]  (11.12)
\[\left( \frac{\partial}{\partial r_o}\right)_t = \left( \frac{\partial}{\partial r_o}\right)_T + \frac{n}{c} \left( \frac{\partial}{\partial T}\right)_{r_o}\]  (11.13)
(d) We see that the radiation (electric field) is proportional to the transverse piece of the $\partial_t J$

$$-n \times (n \times \partial_t J) = \partial_t J - n(n \cdot \partial_t J)$$

(11.14)

In general the transverse projection of a vector is

$$-n \times (n \times V) = V - n(n \cdot V)$$

(11.15)

(e) Power radiated per solid angle is for $r \to \infty$ is

$$\frac{dW}{dt dΩ} = \frac{dP(t)}{dΩ} = \text{energy per observation time per solid angle}$$

and

$$\frac{dP(t)}{dΩ} = r^2 S \cdot n$$

(11.16)

$$= c |r E|^2$$

(11.17)

$$= c |r E|^2$$

(11.18)

11.2 Examples of Non-relativistic Radiation: L31

In this section we will derive several examples of radiation in non-relativistic systems. In a non-relativistic approximation

$$T = t - \frac{r}{c} + \frac{n}{c} \cdot r_o$$

(11.19)

The underlined terms are small: If the typical time and size scales of the source are $T_{\text{typ}}$ and $L_{\text{typ}}$, then $t \sim T_{\text{typ}}$, and $r_o \sim L_{\text{typ}}$, and the ratio the underlined term to the leading term is:

$$\frac{L_{\text{typ}}}{cT_{\text{typ}}} \ll 1$$

(11.20)

This is the non-relativistic approximation. For a harmonic time dependence, $1/T_{\text{typ}} \sim \omega_{\text{typ}}$, and this says that the wave number $k = \frac{2\pi}{\lambda}$ is small compared to the size of the source, i.e. the wavelength of the emitted light is long compared to the size of the system in non-relativistic motion:

$$\frac{2\pi L_{\text{typ}}}{\lambda} \ll 1$$

(11.21)

(a) Keeping only $t - r/c$ and dropping all powers of $n \cdot r_o/c$ in $T$ results in the electric dipole approximation, and also the Larmour formula.

(b) Keeping the first order terms in

$$\frac{n}{c} \cdot r_o$$

(11.22)

results in the magnetic dipole and quadrupole approximations.

The Larmour Formula

(a) For a particle moves slowly with velocity and acceleration, $\mathbf{v}(t)$ and $\mathbf{a}(t)$ along a trajectory $r_*(t)$

(b) We make an ultimate non-relativistic approximation for $T$

$$T \simeq t - \frac{r}{c} \equiv t_e$$

(11.23)

Then we derived the radiation field by substituting the current

$$\mathbf{J}(t_e) = ev(t_e)\delta^3(r_o - r_*(t_e))$$

(11.24)

into the Eqs. (11.8),(11.9), and (11.17) for the radiated power
(c) The electric field is
\[ E = \frac{e}{4\pi rc^2} n \times n \times a(t_e) \]  
(11.25)

Notice that the electric field is of order
\[ E \sim \frac{e}{4\pi r} \frac{a(t_e)}{c^2} \]  
(11.26)

(d) The power per solid angle emitted by acceleration at time \( t_e \) is
\[ \frac{dP(t_e)}{d\Omega} = \frac{e^2}{(4\pi)^2c^3} a^2(t_e) \sin^2 \theta \]  
(11.27)

Notice that the power is of order
\[ P \sim c|E|^2 \sim \frac{a^2}{c^3} \]  
(11.28)

(e) The total energy that is emitted is
\[ P(t_e) = \frac{e^2}{4\pi} \frac{2a^2(t_e)}{3c^4} \]  
(11.29)

**The Electric Dipole approximation**

(a) We make the ultimate non-relativistic approximation
\[ J(t - \frac{r}{c} + \frac{n \cdot r_o}{c}) \simeq J(t - \frac{r}{c}) \]  
(11.30)

Leading to an expression for \( A_{rad} \)
\[ A_{rad} = \frac{1}{4\pi r c} \frac{1}{c} \partial_t p(t_e) \]  
(11.31)

where the dipole moment is
\[ p(t_e) = \int d^3x_o \rho(t_e) r_o \]  
(11.32)

(b) The electric and magnetic fields are
\[ E_{rad} = n \times n \times \frac{1}{c} \partial_t A_{rad} \]  
\[ = \frac{1}{4\pi rc^2} n \times n \times \ddot{p}(t_e) \]  
(11.33)

\[ B_{rad} = n \times E_{rad} \]  
(11.34)

(c) The power radiated is
\[ \frac{dP(t_e)}{d\Omega} = \frac{1}{16\pi^2} \frac{\dot{p}^2(t_e)}{c^3} \sin^2 \theta \]  
(11.36)

(d) For a harmonic source \( p(t_e) = p_o e^{-i\omega(t-r/c)} \) the time averaged power is
\[ P = \frac{1}{4\pi} \frac{\omega^4}{3c^3} |p_o|^2 \]  
(11.37)
The magnetic dipole and quadrupole approximation: L32

(a) In the magnetic dipole and quadrupole approximation we expand the current

\[
J(T) \simeq J(t_e) + \frac{n \cdot r_o}{c} \partial_t J(t_e, r_o)/c
\]

(11.38)

The next term when substituted into Eq. (11.8) gives rise to two new contributions to \(A_{\text{rad}}\), the magnetic dipole and electric quadrupole terms:

\[
A_{\text{rad}} = A_{\text{rad}}^{E1} + A_{\text{rad}}^{M1} + A_{\text{rad}}^{E2}
\]

(11.39)

(b) The magnetic dipole contribution gives

\[
A_{\text{rad}}^{M1} = -\frac{1}{4\pi r} \frac{n}{c} \times \dot{m}(t_e)
\]

(11.40)

where \(m\)

\[
m \equiv \frac{1}{2} \int_{r_o} \rho \times J(t_e, r_o)/c,
\]

(11.41)

is the magnetic dipole moment.

(c) The structure of magnetic dipole radiation is very similar to electric dipole radiation with the duality transformation

E-dipole \quad \rightarrow \quad M-dipole

\[
p \quad \rightarrow \quad m
\]

(11.42)

E \quad \rightarrow \quad B

(11.43)

B \quad \rightarrow \quad -E

(11.44)

(d) The power is

\[
dP_{M1}(t_e) = \frac{\dot{m}^2 \sin^2 \theta}{16\pi^2 c^3}
\]

(11.45)

(e) The power radiated in magnetic dipole radiation is smaller than the power radiated in electric dipole radiation by a factor of the typical velocity, \(v_{\text{typ}}\) squared:

\[
\frac{P_{M1}}{P_{E1}} \propto \frac{m^2}{p^2} \sim \left(\frac{v_{\text{typ}}}{c}\right)^2
\]

(11.46)

where \(v_{\text{typ}} \sim L_{\text{typ}}/T_{\text{typ}}\).

Quadrupole radiation

(a) For quadrupole radiation we have

\[
A_{\text{rad}}^{E2} = \frac{1}{24\pi r} \frac{n_i}{c^2} \tilde{Q}^{ij}
\]

(11.48)

where \(Q^{ij}\) is the symmetric traceless quadrupole tensor.

\[
Q^{ij} = \int d^3x_o \rho(t_e, r_o) \left(3r_o^i r_o^j - r_o^2 \delta^{ij}\right)
\]

(11.49)

(b) The electric field is

\[
E_{\text{rad}} = \frac{-1}{24\pi r c^3} \left[\tilde{Q} \cdot n - n(n^\top \cdot \tilde{Q} \cdot n)\right]
\]

(11.50)

where (more precisely) the first term in square brackets means \(n_i \tilde{Q}^{ij} n_j\), while the second term means, \((n_i \tilde{Q}^{im} n_m)n_j\).
(c) A fair bit of algebra shows that the total power radiated from a quadrupole form is

\[ P = \frac{1}{720 \pi c^5} Q^{ab} Q_{ab} \]  

(11.51)

(d) For harmonic fields, \( Q = Q_o e^{-i\omega t} \), the time averaged power is rises as \( \omega^6 \)

\[ P = \frac{c}{1440 \pi} \left( \frac{\omega}{c} \right)^6 Q_o^2 \]  

(11.52)

(e) The total power radiated radiated in quadrupole radiation to electric-dipole radiation for a typical source size \( L_{\text{typ}} \) is smaller:

\[ \frac{P_{E2}}{P_{E1}} \sim \left( \frac{\omega L_{\text{typ}}}{c} \right)^2 \]  

(11.53)

11.3 ATENAS

(a) In an antenna with sinusoidal frequency we have

\[ \mathbf{J}(T, r_o) = e^{-i\omega(t - \frac{r}{c} + \frac{n \cdot r_o}{c})} \mathbf{J}(r_o) \]  

(11.54)

(b) Then the radiation field for a sinusoidal current is:

\[ \mathbf{A}_{\text{rad}} = \frac{e^{-i\omega(t - r/c)}}{4\pi r} \int_{r_o} e^{-i\omega \frac{n \cdot r_o}{c}} \mathbf{J}(r_o) / c \]  

(11.55)

In general one will need to do this integral to determine the radiation field.

(c) The typical radiation resistance associated with driving a current which will radiate over a wide range of frequencies is \( R_{\text{vacuum}} = c \mu_o = \sqrt{\mu_o / \epsilon_o} = 376 \text{ Ohm} \).