11 Radiation in Non-relativistic Systems

11.1 Basic equations

This first section will *NOT* make a non-relativistic approximation, but will examine the far field limit.

(a) We wrote down the wave equations in the covariant gauge:

$$-\Box\Phi = \rho(t_o, \boldsymbol{r}_o) \tag{11.1}$$
$$-\Box\boldsymbol{A} = \boldsymbol{J}(t_o, \boldsymbol{r}_o)/c \tag{11.2}$$

The gauge condition reads

$$\frac{1}{c}\partial_t\Phi + \nabla\cdot\boldsymbol{A} = 0 \tag{11.3}$$

(b) Then we used the green function of the wave equation

$$G(t, r|t_o r_o) = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\delta(t - t_o + \frac{|\boldsymbol{r}-\boldsymbol{r}_o|}{c}) \tag{11.4}$$

to determine the potentials $(\Phi, \boldsymbol{A})$

$$\Phi(t,\boldsymbol{r}) = \int \mathrm{d}^3x_o \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\rho(T, \boldsymbol{r}_o) \tag{11.5}$$
$$\boldsymbol{A}(t,\boldsymbol{r}) = \int \mathrm{d}^3x_o \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\boldsymbol{J}(T, \boldsymbol{r}_o)/c \tag{11.6}$$

Here $T(t, \boldsymbol{r})$ is the retarded time

$$T(t,r) = t - \frac{|\boldsymbol{r}-\boldsymbol{r}_o|}{c} \tag{11.7}$$

(c) We used the potentials to determine the electric and magnetic fields. Electric and magnetic fields in the far field are

$$\boldsymbol{A}_{\mathrm{rad}}(t,\boldsymbol{r}) = \frac{1}{4\pi r}\int_{\boldsymbol{r}_o} \frac{\boldsymbol{J}(T,\boldsymbol{r}_o)}{c} \tag{11.8}$$

and

$$\boldsymbol{B}(t,\boldsymbol{r}) = -\frac{\boldsymbol{n}}{c}\times\partial_t\boldsymbol{A}_{\mathrm{rad}} \tag{11.9}$$
$$\boldsymbol{E}(t,\boldsymbol{r}) = \boldsymbol{n}\times\frac{\boldsymbol{n}}{c}\times\partial_t\boldsymbol{A}_{\mathrm{rad}} = -\boldsymbol{n}\times\boldsymbol{B}(t,\boldsymbol{r}) \tag{11.10}$$

In the far field (large distance limit $\boldsymbol{r}\to\infty$) limit we have

$$T = t - \frac{r}{c} + \boldsymbol{n}\cdot\frac{\boldsymbol{r}_o}{c} \tag{11.11}$$

And we recording the derivatives

$$\left(\frac{\partial}{\partial t}\right)_{r_o} = \left(\frac{\partial}{\partial T}\right)_{r_o} \tag{11.12}$$
$$\left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_t = \left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_T + \frac{\boldsymbol{n}}{c}\left(\frac{\partial}{\partial T}\right)_{\boldsymbol{r}_o} \tag{11.13}$$

(d) We see that the radiation (electric field) is proportional to the transverse piece of the $\partial_t \boldsymbol{J}$

$$-\boldsymbol{n} \times (\boldsymbol{n} \times \partial_t \boldsymbol{J}) = \partial_t \boldsymbol{J} - \boldsymbol{n}(\boldsymbol{n} \cdot \partial_t \boldsymbol{J}) \tag{11.14}$$

In general the transverse projection of a vector is

$$-\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{V}) = \boldsymbol{V} - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{V}) \tag{11.15}$$

(e) Power radiated per solid angle is for $r \to \infty$ is

$$\frac{dW}{dtd\Omega} = \frac{dP(t)}{d\Omega} = \text{energy per observation time per solid angle} \tag{11.16}$$

and

$$\frac{dP(t)}{d\Omega} = r^2 \boldsymbol{S} \cdot n \tag{11.17}$$

$$= c|rE|^2 \tag{11.18}$$

11.2 Examples of Non-relativistic Radiation: L31

In this section we will derive several examples of radiation in non-relativistic systems. In a non-relativistic approximation

$$T = t - \frac{r}{c} + \underbrace{\frac{\boldsymbol{n}}{c} \cdot \boldsymbol{r}_o}_{\text{small}} \tag{11.19}$$

The underlined terms are small: If the typical time and size scales of the source are $T_{\rm typ}$ and $L_{\rm typ}$, then $t \sim T_{\rm typ}$, and $\boldsymbol{r}_o \sim L_{\rm typ}$, and the ratio the underlined term to the leading term is:

$$\frac{L_{\rm typ}}{cT_{\rm typ}} \ll 1 \tag{11.20}$$

This is the non-relativistic approximation. For a harmonic time dependence, $1/T_{\rm typ} \sim \omega_{\rm typ}$, and this says that the wave number $k = \frac{2\pi}{\lambda}$ is small compared to the size of the source, *i.e. the wave length of the emitted light is long compared to the size of the system in non-relativistic motion:*

$$\frac{2\pi L_{\rm typ}}{\lambda} \ll 1 \tag{11.21}$$

(a) Keeping only $t - r/c$ and dropping all powers of $\boldsymbol{n} \cdot \boldsymbol{r}_o / c$ in T results in the electric dipole approximation, and also the Larmour formula.

(b) Keeping the first order terms in

$$\frac{\boldsymbol{n}}{c} \cdot \boldsymbol{r}_o \tag{11.22}$$

results in the magnetic dipole and quadrupole approximations.

The Larmour Formula

(a) For a particle moves slowly with velocity and acceleration, $\boldsymbol{v}(t)$ and $\boldsymbol{a}(t)$ along a trajectory $\boldsymbol{r}_*(t)$

(b) We make an ultimate non-relativistic approximation for T

$$T \simeq t - \frac{r}{c} \equiv t_e \tag{11.23}$$

Then we derived the radiation field by substituting the current

$$\boldsymbol{J}(t_e) = e\boldsymbol{v}(t_e)\delta^3(\boldsymbol{r}_o - \boldsymbol{r}_*(t_e)) \tag{11.24}$$

into the Eqs. (11.8),(11.9), and (11.17) for the radiated power

(c) The electric field is

$$\boldsymbol{E} = \frac{e}{4\pi r c^2}\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{a}(t_e) \tag{11.25}$$

Notice that the electric field is of order

$$E \sim \frac{e}{4\pi r}\,\frac{a(t_e)}{c^2} \tag{11.26}$$

(d) The power per solid angle emitted by acceleration at time t_e is

$$\frac{dP(t_e)}{d\Omega} = \frac{e^2}{(4\pi)^2 c^3} a^2(t_e)\sin^2\theta \tag{11.27}$$

Notice that the power is of order

$$P \sim c|rE|^2 \sim \frac{a^2}{c^3} \tag{11.28}$$

(e) The total energy that is emitted is

$$P(t_e) = \frac{e^2}{4\pi}\frac{2}{3}\frac{a^2(t_e)}{c^3} \tag{11.29}$$

The Electric Dipole approximation

(a) We make the ultimate non-relativistic approximation

$$\boldsymbol{J}(t - \frac{r}{c} + \frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c}) \simeq \boldsymbol{J}(t-\frac{r}{c}) \tag{11.30}$$

Leading to an expression for $\boldsymbol{A}_{\rm rad}$

$$\boldsymbol{A}_{\rm rad} = \frac{1}{4\pi r}\frac{1}{c}\partial_t \boldsymbol{p}(t_e) \tag{11.31}$$

where the dipole moment is

$$\boldsymbol{p}(t_e) = \int \mathrm{d}^3x_o\, \rho(t_e)\boldsymbol{r}_o \tag{11.32}$$

(b) The electric and magnetic fields are

$$\boldsymbol{E}_{\rm rad} = \boldsymbol{n}\times\boldsymbol{n}\times\frac{1}{c}\partial_t\boldsymbol{A}_{\rm rad} \tag{11.33}$$

$$= \frac{1}{4\pi r c^2}\,\boldsymbol{n}\times\boldsymbol{n}\times\ddot{\boldsymbol{p}}(t_e) \tag{11.34}$$

$$\boldsymbol{B}_{\rm rad} = \boldsymbol{n}\times\boldsymbol{E}_{\rm rad} \tag{11.35}$$

(c) The power radiated is

$$\frac{dP(t_e)}{d\Omega} = \frac{1}{16\pi^2}\frac{\ddot{p}^2(t_e)}{c^3}\sin^2\theta \tag{11.36}$$

(d) For a harmonic source $\boldsymbol{p}(t_e) = \boldsymbol{p}_o e^{-i\omega(t-r/c)}$ the time averaged power is

$$P = \frac{1}{4\pi}\frac{\omega^4}{3c^3}|\boldsymbol{p}_o|^2 \tag{11.37}$$

The magnetic dipole and quadrupole approximation: L32

(a) In the magnetic dipole and quadrupole approximation we expand the current

$$\boldsymbol{J}(T) \simeq \underbrace{\boldsymbol{J}(t_e)}_{\text{electric dipole}} + \underbrace{\frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c}\partial_t \boldsymbol{J}(t_e,\boldsymbol{r}_o)/c}_{\text{next term}} \tag{11.38}$$

The next term when substituted into Eq. (11.8) gives rise two new contributions to $\boldsymbol{A}_{\rm rad}$, the magnetic dipole and electric quadrupole terms:

$$\boldsymbol{A}_{\rm rad} = \underbrace{\boldsymbol{A}^{E1}_{\rm rad}}_{\text{electric dipole}} + \underbrace{\boldsymbol{A}^{M1}_{\rm rad}}_{\text{mag dipole}} + \underbrace{\boldsymbol{A}^{E2}_{\rm rad}}_{\text{electric-quad}} \tag{11.39}$$

(b) The magnetic dipole contribution gives

$$\boldsymbol{A}^{M1}_{\rm rad} = \frac{-1}{4\pi r}\frac{\boldsymbol{n}}{c}\times \dot{\boldsymbol{m}}(t_e) \tag{11.40}$$

where $\boldsymbol{m}$

$$\boldsymbol{m} \equiv \frac{1}{2}\int_{\boldsymbol{r}_o} \boldsymbol{r}_o \times \boldsymbol{J}(t_e,\boldsymbol{r}_o)/c\,, \tag{11.41}$$

is the magnetic dipole moment.

(c) The structure of magnetic dipole radiation is very similar to electric dipole radiation with the duality transformation

$$\text{E-dipole} \to \text{M-dipole} \tag{11.42}$$
$$\boldsymbol{p} \to \boldsymbol{m} \tag{11.43}$$
$$\boldsymbol{E} \to \boldsymbol{B} \tag{11.44}$$
$$\boldsymbol{B} \to -\boldsymbol{E} \tag{11.45}$$

(d) The power is

$$\frac{dP^{M1}(t_e)}{d\Omega} = \frac{\ddot{m}^2\sin^2\theta}{16\pi^2c^3} \tag{11.46}$$

(e) The power radiated in magnetic dipole radiation is smaller than the power radiated in electric dipole radiation by a factor of the typical velocity, $v_{\rm typ}$ squared:

$$\frac{P^{M1}}{P^{E1}} \propto \frac{m^2}{p^2} \sim \left(\frac{v_{\rm typ}}{c}\right)^2 \tag{11.47}$$

where $v_{\rm typ} \sim L_{\rm typ}/T_{\rm typ}$

Quadrupole rdiation

(a) For quadrupole radiation we have

$$\boldsymbol{A}^j_{\rm rad,E2} = \frac{1}{24\pi r}\frac{n_i}{c^2}\dddot{\mathcal{Q}}^{ij} \tag{11.48}$$

where $\mathcal{Q}^{ij}$ is the symmetric traceless quadrupole tensor.

$$\mathcal{Q}^{ij} = \int \mathrm{d}^3x_o\rho(t_e,\boldsymbol{r}_o)\,\left(3r_o^ir_o^j - \boldsymbol{r}_o^2\delta^{ij}\right) \tag{11.49}$$

(b) The electric field is

$$\boldsymbol{E}_{\rm rad} = \frac{-1}{24\pi rc^3}\left[\dddot{\boldsymbol{\mathcal{Q}}}\cdot\boldsymbol{n} - \boldsymbol{n}(\boldsymbol{n}^\top\cdot\dddot{\boldsymbol{\mathcal{Q}}}\cdot\boldsymbol{n})\right] \tag{11.50}$$

where (more precisely) the first term in square brackets means $n_i\dddot{\mathcal{Q}}^{ij}$, while the second term means, $(n_\ell\dddot{\mathcal{Q}}^{\ell m}n_m)n^j$.

(c) A fair bit of algebra shows that the total power radiated from a quadrupole form is

$$P = \frac{1}{720\pi c^5} \dddot{\mathcal{Q}}^{ab} \dddot{\mathcal{Q}}_{ab} \tag{11.51}$$

(d) For harmonic fields, $\mathcal{Q} = \mathcal{Q}_o e^{-i\omega t}$, the time averaged power is rises as ω^6

$$P = \frac{c}{1440\pi} \left(\frac{\omega}{c}\right)^6 \mathcal{Q}_o^2 \tag{11.52}$$

(e) The total power radiated radiated in quadrupole radiation to electric-dipole radiation for a typical source size $L_{\rm typ}$ is smaller:

$$\frac{P^{E2}}{P^{E1}} \sim \left(\frac{\omega L_{\rm typ}}{c}\right)^2 \tag{11.53}$$

11.3 Attenas

(a) In an antenna with sinusoidal frequency we have

$$\boldsymbol{J}(T, \boldsymbol{r}_o) = e^{-i\omega(t - \frac{r}{c} + \frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c})} \boldsymbol{J}(\boldsymbol{r}_o) \tag{11.54}$$

(b) Then the radiation field for a sinusoidal current is:

$$\boldsymbol{A}_{\rm rad} = \frac{e^{-i\omega(t-r/c)}}{4\pi r} \int_{\boldsymbol{r}_o} e^{-i\omega\frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c}} \boldsymbol{J}(\boldsymbol{r}_o)/c \tag{11.55}$$

In general one will need to do this integral to determine the radiation field.

(c) The typical radiation resistance associated with driving a current which will radiate over a wide range of frequencies is $R_{\rm vacuum} = c\mu_o = \sqrt{\mu_o/\epsilon_o} = 376\,\text{Ohm}$.