Covariant Electrodynamics

① $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu} = \left(\dfrac{1}{c}\dfrac{\partial}{\partial t}, \vec{\nabla} \right)$ transforms as a four vector

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\mu}\left(L^\mu{}_\nu\right)$$

There is also contravariant components $\partial^\mu = \left(-\dfrac{1}{c}\partial_t, \vec{\nabla}\right) = \dfrac{\partial}{\partial x_\mu}$

So

$$\partial_\mu \partial^\mu = -\frac{1}{c^2}\frac{\partial}{\partial t^2} + \nabla^2 \equiv \Box$$

is invariant

② Then there is the continuity Eqn

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{j} = 0 \Rightarrow \frac{1}{c}\frac{\partial}{\partial t}(c\rho) + \nabla\cdot J = 0$$

So take, $J^\mu = (c\rho, \vec{J})$, as a four vector,

$$\partial_\mu J^\mu = 0$$

③ Then the equations for the gauge potential

$$-\Box\varphi = J^0/c$$
$$-\Box\vec{A} = \vec{J}/c$$

Together with the lorentz gauge condition:

$$\frac{1}{c}\frac{\partial}{\partial t}\varphi + \nabla\cdot\vec{A} = 0$$

So in order to have a lorentz invariant theory take $(\varphi, \vec{A})$ to be a four vector

$$A^\mu = (\varphi, \vec{A})$$

Then the wave eqn becomes

$$\boxed{-\Box A^\mu = J^\mu / c} \quad \text{and} \quad \boxed{\partial_\mu A^\mu = 0}$$

(4) Now the fields

$$\left.\begin{aligned} \vec{E} &= -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi \\ \vec{B} &= \nabla \times \vec{A} \end{aligned}\right\}$$

These are combined into a rank 2 antisymmetric tensor

where

$$F^{\alpha\beta} = \begin{array}{c|c} & \beta \longrightarrow \\ \hline \alpha \downarrow & \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix} \end{array} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

We can see this

$$F^{0i} \equiv E^i = -\frac{1}{c}\frac{\partial A^i}{\partial t} - \frac{\partial \varphi}{\partial x_i} = \partial^0 A^i - \partial^i A^0 \equiv F^{0i}$$

$$B_k = (\nabla \times A)_k$$

$$F^{ij} = \varepsilon^{ijk} B_k = \underbrace{\varepsilon^{ijk}\varepsilon_{klm}}\partial^l A^m$$

$$(\delta^i_l \delta^j_m - \delta^j_l \delta^k_m)\partial^l A^m$$

$$= \partial^i A^j - \partial^j A^i$$

So $F^{\alpha\beta} = \partial^\alpha A - \partial^\beta A^\alpha$ transforms as a second rank tensor in the following way

$$F^{\mu\nu} = L^\mu{}_\alpha L^\nu{}_\beta F^{\alpha\beta}$$

Excercise,

- Show that

$$F^i = F^{0i} = -F^{i0} = F^i{}_0 = -F_0{}^i = F^{0i}$$

(5) Now the EOM (Part I)

$$\nabla \cdot E = \rho \implies -\partial_\ell \underbrace{F^{\ell 0}}_{(-E^\ell)} = \underbrace{J^0/c}_{\rho}$$

and

$$-\frac{1}{c}\partial_t E + \nabla \times B = \frac{J}{c} \implies -\left(\frac{\partial F^{0i}}{\partial x^0} + \frac{\partial F^{\ell i}}{\partial x^\ell}\right) \doteq J^i/c$$

So

$$\boxed{-\partial_\alpha F^{\alpha\beta} = \frac{J^\beta}{c}}$$

Excercise:

- Starting from $-\partial_\alpha F^{\alpha\beta} = J^\beta/c$ and the definition of $F^{\alpha\beta}$ derive,

$$-\Box A^\beta = J^\beta/c$$

Solution

Lorentz Gauge

$$-\partial_\alpha \underbrace{(\partial^\alpha A^\beta - \partial^\beta A^\alpha)}_{\equiv F^{\alpha\beta}} = -\partial_\alpha \partial^\alpha A^\beta + \partial^\beta(\partial_\alpha A^\alpha) = J^\beta$$

Lorentz Gauge $\partial_\alpha A^\alpha = 0$, so

$$-\partial_\alpha \partial^\alpha A^\beta = 0 \quad \text{or} \quad -\Box A^\beta = J^\beta$$

⑥ The Remaing Maxwell Eqs

$$\nabla \cdot B = 0$$

$$\frac{1}{c} \partial_t \vec{B} + \nabla \times E = 0$$

Comparison with the first two eqs in absence of currents gives

$$\left.\begin{aligned} \nabla \cdot E &= 0 \\ -\frac{1}{c}\frac{\partial E}{\partial t} + \nabla \times B &= 0 \end{aligned}\right\}$$

So the second two maxwell eqs involve the replacement (duality) $E \to B$ and $B \to -E$.

Thus define the dual tensor

$$\tilde{F}^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & B^x & B^y & B^z \\ \hline -B^x & 0 & -E^z & E^y \\ -B^y & E^z & 0 & -E^x \\ -B^z & -E^y & E^x & 0 \end{array}\right)$$

So

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

The dual tensor can be defined from $F_{\mu\nu}$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$ ← this implements the replacement $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$

Here

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{for even perms of } 0,1,2,3 \\ -1 & \text{for odd perms of } 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

Expressing in therms of

antisymmetric in $\mu\alpha\beta$

$$\partial_\mu \tilde{F}^{\mu\nu} = -\frac{1}{2} \varepsilon^{\nu\mu\alpha\beta} \partial_\mu F_{\alpha\beta} = 0$$

This can be written as the Bianchi-Identity

$$\boxed{\partial_{[\mu_1} F_{\mu_2\mu_3]} = 0} \quad \text{or} \quad \partial_{\mu_1} F_{\mu_2\mu_3} - \partial_{\mu_2} F_{\mu_1\mu_3} + \partial_{\mu_3} F_{\mu_1\mu_2}$$

where $\partial_{[\mu_1} F_{\mu_2\mu_3]}$ stands for the antisymmetric combo

examples

like a 2×2 determinant

$$T_{[\mu_1\mu_2]} = \frac{1}{2!}\left(T_{\mu_1\mu_2} - T_{\mu_2\mu_1}\right)$$

like a 3x3 determ

$$T_{[\mu_1\mu_2\mu_3]} = \frac{1}{3!}\Big[\left(T_{\mu_1\mu_2\mu_3} - T_{\mu_1\mu_3\mu_2}\right) - \left(T_{\mu_2\mu_1\mu_3} - T_{\mu_2\mu_3\mu_1}\right) + \left(T_{\mu_3\mu_1\mu_2} - T_{\mu_3\mu_2\mu_1}\right)\Big]$$

Excercise

Show that is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ then the second two Maxwell eqs are altomatically satified

Solutión

$$\partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

$$= -\frac{1}{2} \varepsilon^{\nu\mu\alpha\beta} (\partial_\mu \partial_\alpha A_\beta - \partial_\mu \partial_\beta A_\alpha) = 0$$

But, $\underbrace{\partial_\mu \partial_\alpha A_\beta = \partial_\alpha \partial_\mu A_\beta}_{\text{Symmetric}}$ and $\underbrace{\varepsilon^{\nu\mu\alpha\beta} = -\varepsilon^{\nu\alpha\mu\beta}}_{\text{antisymmetric}}$

And the contraction of antisymm and a symmetric tensor gives zero.

Last Time

- Finished by discussing the stress tensor:

$$\Theta^{\mu\nu}_{TOT} = \left(\begin{array}{c|c} u_{TOT} & \vec{S}_{tot}/c \\ \hline c\vec{g}_{Tot} & T^{ij} \end{array}\right) \quad \text{wih} \quad \partial_\mu \Theta^{\mu\nu}_{TOT} = 0$$

E-consv

0-component

$$\left.\begin{array}{l} \Theta^{00}_{TOT} = \text{energy density} = u_{TOT} \\ \Theta^{0i}_{TOT} = \text{energy flux} = \vec{S}/c = \vec{g}c \end{array}\right\} \quad \partial_\mu \Theta^{\mu 0}_{TOT} = 0$$

Mom-consv

~~i-th component~~ of momentum

$$\left.\begin{array}{l} \Theta^{i0}_{TOT} = \text{momentum density} = \vec{g}c = \vec{S}/c \\ \Theta^{ij}_{TOT} = \text{stress force/area} = T^{ij} \end{array}\right\} \quad \partial_\mu \Theta^{\mu i} = 0$$

If I have a mechanical system (like a fluid) with currents then the E+M fields will push and pull the system:

$$\partial_\mu \Theta^{\mu\nu}_{mech} = \underbrace{F^{\nu}{}_{\rho} \frac{J^\rho}{c}}_{\text{four force}}$$

$$\partial_\mu \Theta^{\mu 0}_{mech} = \vec{E} \cdot \frac{\vec{J}}{c}$$

$$\partial_\mu \Theta^{\mu i}_{mech} = \rho E^i + \left(\frac{J}{c} \times B\right)^i$$

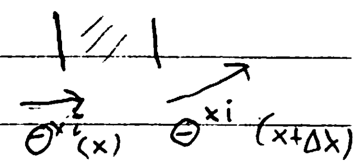
And thus mechanical energy and momentum won't be conserved.

Last Time

The electromagnetic force must be the divergence of something:

differences of force/area

$$F^{\nu}{}_{\rho}\frac{J^{\rho}}{c} = -\partial_{\mu}\Theta^{\mu\nu}_{em}$$



Homework, show using $-\partial_{\mu}F^{\mu\nu} = J^{\nu}/c$ that

$$\Theta^{\mu\nu}_{em} = F^{\mu\lambda}F^{\nu}{}_{\lambda} + g^{\mu\nu}\left(-\frac{1}{4}F^2\right)$$ (see below)

Then

$$\partial_{\mu}\Theta^{\mu\nu}_{mech} = -\partial_{\mu}\Theta^{\mu\nu}_{em}$$

or

$$\partial_{\mu}\left(\Theta^{\mu\nu}_{mech} + \Theta^{\mu\nu}_{em}\right) = 0$$

and thus the combined mechanical + electromagnetic energy and momentum will be conserved.

$$\Theta^{\mu\nu}_{em} = \left(\begin{array}{c|c} \frac{1}{2}(E^2+B^2) & \vec{E}\times\vec{B} \\ \hline \vec{E}\times\vec{B} & \begin{array}{c} -E^iE^j + \frac{1}{2}\delta^{ij}E^2 \\ + \\ -B^iB^j + \frac{1}{2}\delta^{ij}B^2 \end{array} \end{array}\right) = \left(\begin{array}{c|c} u_{em} & \vec{S}_{em}/c \\ \hline \vec{g}_{em}\cdot c & T^{ij}_{em} \end{array}\right)$$