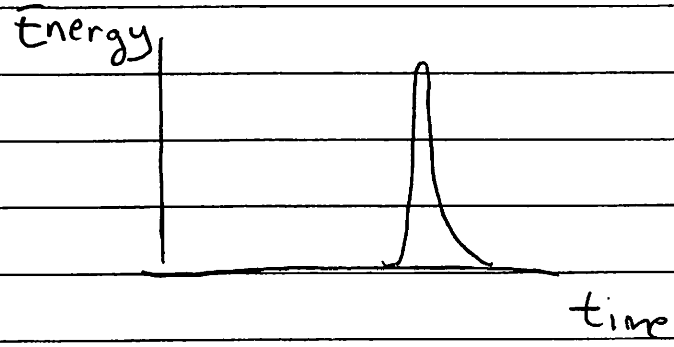
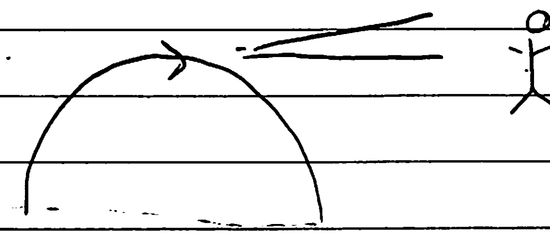
The Fourier Spectrum





$$\frac{dW}{dt\,d\Omega} = c\,|r E_{rad}(t)|^2$$ ← energy per observers time

So

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt\; c\,|r E_{rad}(t)|^2\, dt$$

Using Parsevals Thrm (Proved in Homework #1)

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\; c\,|r E_{rad}(\omega)|^2$$

Where

$$E_{rad}(\omega) = \int_{-\infty}^{\infty} e^{+i\omega t}\, E_{rad}(t)$$

$$E_{rad}(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\, E_{rad}(\omega)$$

So we have that

$$2\pi \frac{dW}{d\omega\, d\Omega} = c\left| r\, E_{rad}(\omega)\right|^2$$

The sign of ω is not physically relavant.
Since $E(t)$ is real $E(-\omega) = E^*(\omega)$.
Thus define (also incorparating a 2π)

$$\frac{dI}{d\omega\, d\Omega} \equiv \frac{c}{2\pi}\left(\left| r E_{rad}(\omega)\right|^2 + \left| r E_{rad}(-\omega)\right|^2 \right)$$

$$\frac{dI}{d\omega\, d\Omega} = \frac{c}{\pi}\left| r E_{rad}(\omega)\right|^2 \qquad \text{with } \omega > 0$$

So that

$$\frac{dW}{d\Omega} = \int_0^\infty \frac{dI}{d\omega\, d\Omega}\, d\omega$$

So the number of photons between ω + $(\omega + d\omega$

$$\hbar\omega \frac{dN}{d\omega\, d\Omega}\, d\omega = \frac{dI}{d\omega\, d\Omega}\, d\omega$$

The Spectrum

$$E(\omega) = \int_{-\infty}^{\infty} dt \; e^{i\omega t} \; E(t)$$

$$= \int_{-\infty}^{\infty} dt \; e^{+i\omega t} \; \frac{q}{4\pi r c^2} \left[\frac{n \times (n-\beta) \times a}{(1 - n\cdot\beta)^3} \right]_{ret}$$

where all variables β and $a(T)$ are supposed to be evaluated at the retarded time. We want to integrate over retarded time

$$T = t - \frac{r}{c} + \frac{n\cdot r_*}{c} \implies t = T + \frac{r}{c} - \frac{\vec{n}\cdot\vec{r}_*}{c}$$

So using

$$\frac{dt}{dT} = 1 - n\cdot\beta$$

$$E(\omega) = e^{i\omega r/c} \int_{-\infty}^{\infty} dT \; \overbrace{(1 - n\cdot\beta)}^{\partial t/\partial T} \; e^{i\omega(T - n\cdot r_o/c)} \; E(T)$$

$$E(\omega) = \frac{q \, e^{i\omega r/c}}{4\pi r c^2} \int_{-\infty}^{\infty} dT \; e^{i\omega(T - n\cdot r_o/c)} \left[\frac{n \times (n-\beta) \times a}{(1 - n\cdot\beta)^2} \right]_{ret}$$

in many ways this form is the simplest. But we can find another form that is often used by recalling that

$$E(T) = n \times n \times \frac{1}{c} \frac{\partial A}{\partial t} = n \times n \times \frac{\partial A}{\partial T} \frac{\partial T}{\partial t}$$

The spectrum pg. 2

So that

$$E(\omega) = e^{i\omega r/c} \int_{-\infty}^{\infty} dT \; e^{i\omega (T - n\cdot r_*/c)} \; \frac{n \times n \times}{c} \frac{\partial A}{\partial T}$$

where, $\vec{A} = \dfrac{q}{4\pi r} \dfrac{\vec{V}(T)/c}{(1 - n\cdot\beta)}$. Integrating by parts, we have using

$$\frac{d}{dT}(T - n\cdot r_*/c) = 1 - n\cdot\beta$$

We have

$$E(\omega) = \frac{q}{4\pi c} e^{i\omega r/c} (-i\omega) \int_{-\infty}^{\infty} dT \, e^{i\omega (T - n\cdot r_*(T))} \, \vec{n} \times \vec{n} \times \vec{\beta}$$

Or

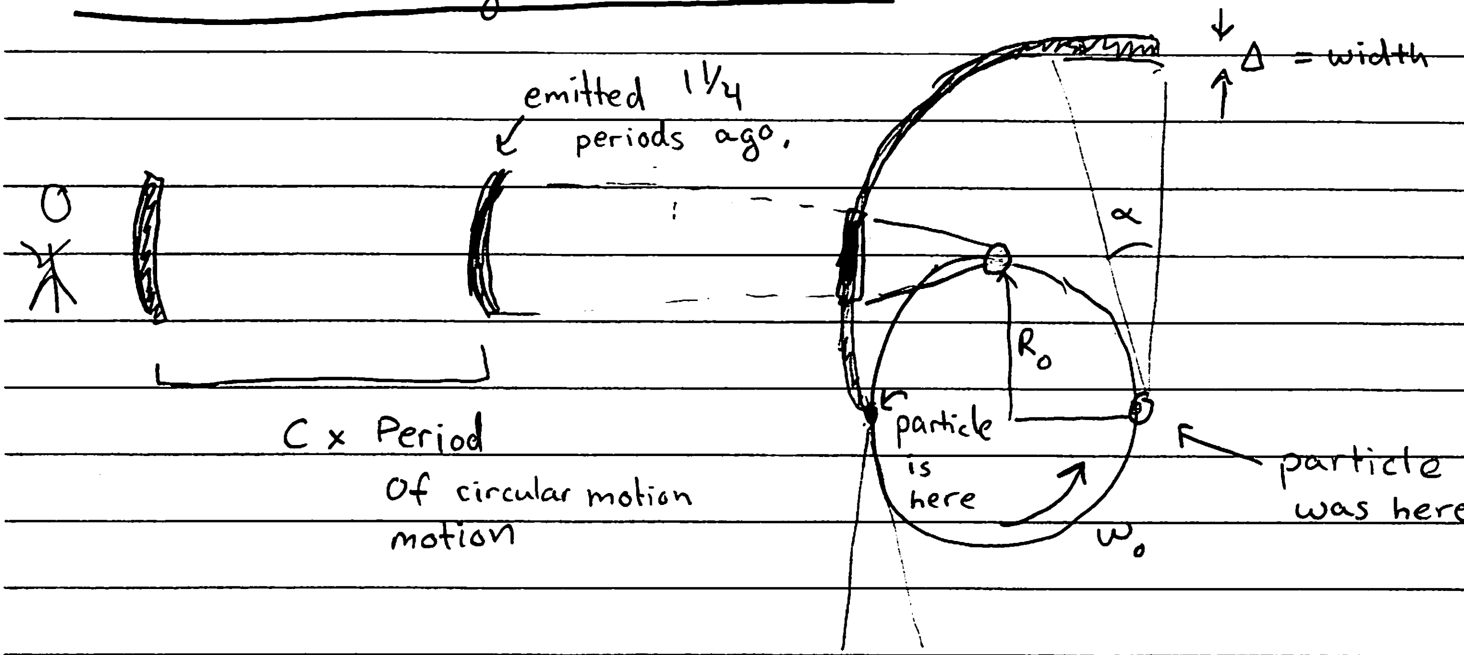
$$\boxed{c\,|E(\omega)|^2 = \frac{q^2}{16\pi^2} \left(\frac{\omega^2}{c^2}\right) \left| \int_{-\infty}^{\infty} dT \, e^{i\omega (T - n\cdot r_*)} \, n \times n \times \beta(T) \right|^2}$$

Eq. (★★★)

This is $2\pi \, dW / d\omega \, d\Omega$. It shows that the electric field is determined by a kind of retarded fourier transform of the transverse current

$$\frac{J_t}{c} = n \times n \times eV/c$$

Radiation During Circular Motion (Synchrotron Radiation)



- Every period the strobelight of the radiation cone points in your direction.

- The pulses of light are short in duration: $\Delta t \sim \Delta/c$ the cone is narrow $\alpha \sim \frac{1}{\gamma}$ (and because of the difference between the formation and observation times ... but that we will discuss below)

- The observer sees a pulse every period:

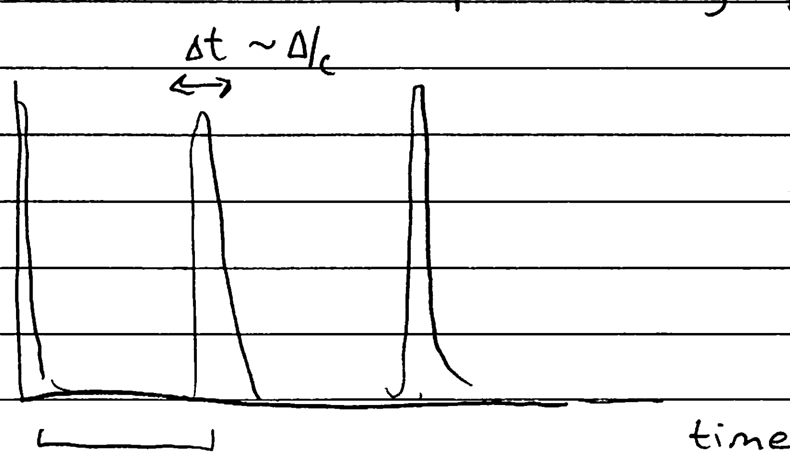


Figure Credit: Christina Athansion et al, arXiv 1001.3880

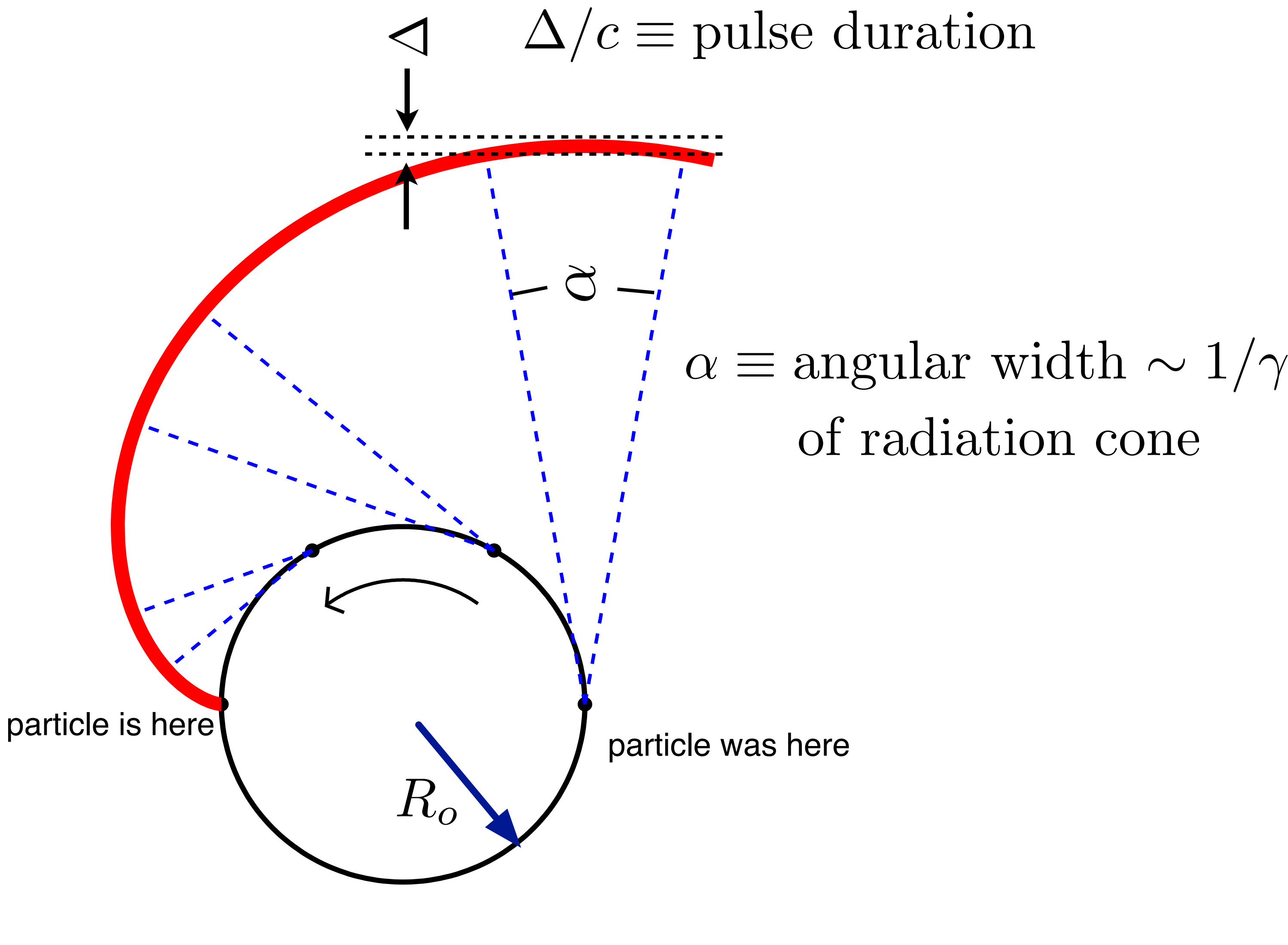
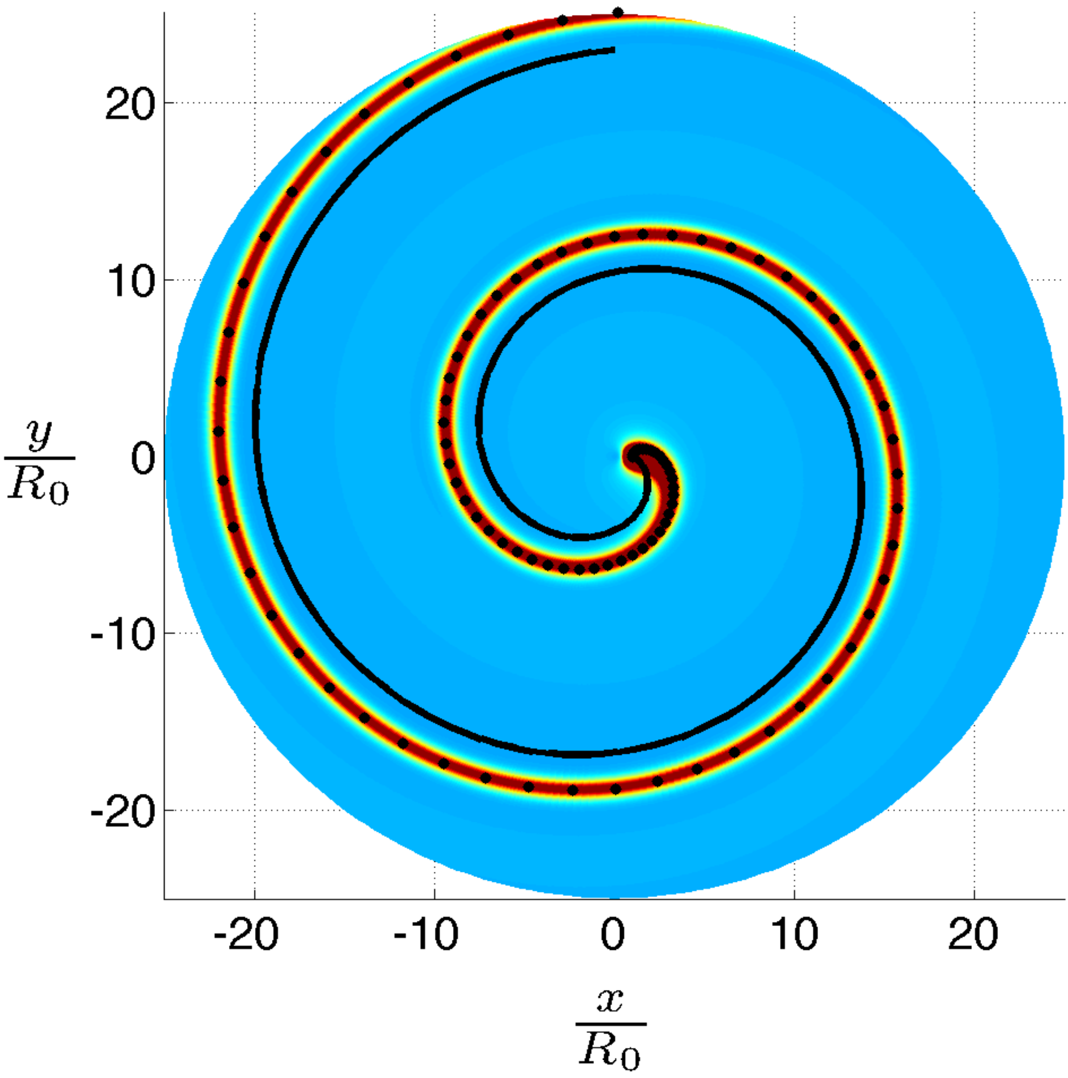


Figure crédit: Christina Athanasion et al, arXiv:1001.3880



Basic Uses of Synchrotron Radiation

- Since the pulse is very narrow in time it contains a wide range of fourier frequencies

$$\Delta W \sim \frac{1}{\Delta t}$$

We should compute the pulse shape, look at its fourier transform and compute the power in each band.

- The light is quite intense
- Both of features are highly desirable

Estimate of The Frequency Width

The frequency width is inversely related to the time width, Δt

$$\Delta W \sim \frac{1}{\Delta t}$$

Before Starting Definitions:

$\alpha \equiv$ angular width of cone, $\alpha \sim 1/\gamma$

$\delta t \equiv \Delta / c \equiv$ duration of pulse = what we want to estimate

Estimate of $\Delta\omega$ pg. 2

See figures!

① At time T_1 at the saurce (retarded time) the spotlight is starting to point in your angular direction. The leading front is emitted

② The strobelight will point in your direction for a time set by the angular width of radiation cone $\alpha \sim \frac{1}{\gamma}$, and the angular velocity:

$$\Delta T = T_2 - T_1 = \frac{\alpha}{\omega_0} = \frac{R_0 \alpha}{V} \sim \frac{R_0}{\gamma c}$$

$\omega_0 = R_0 / V$

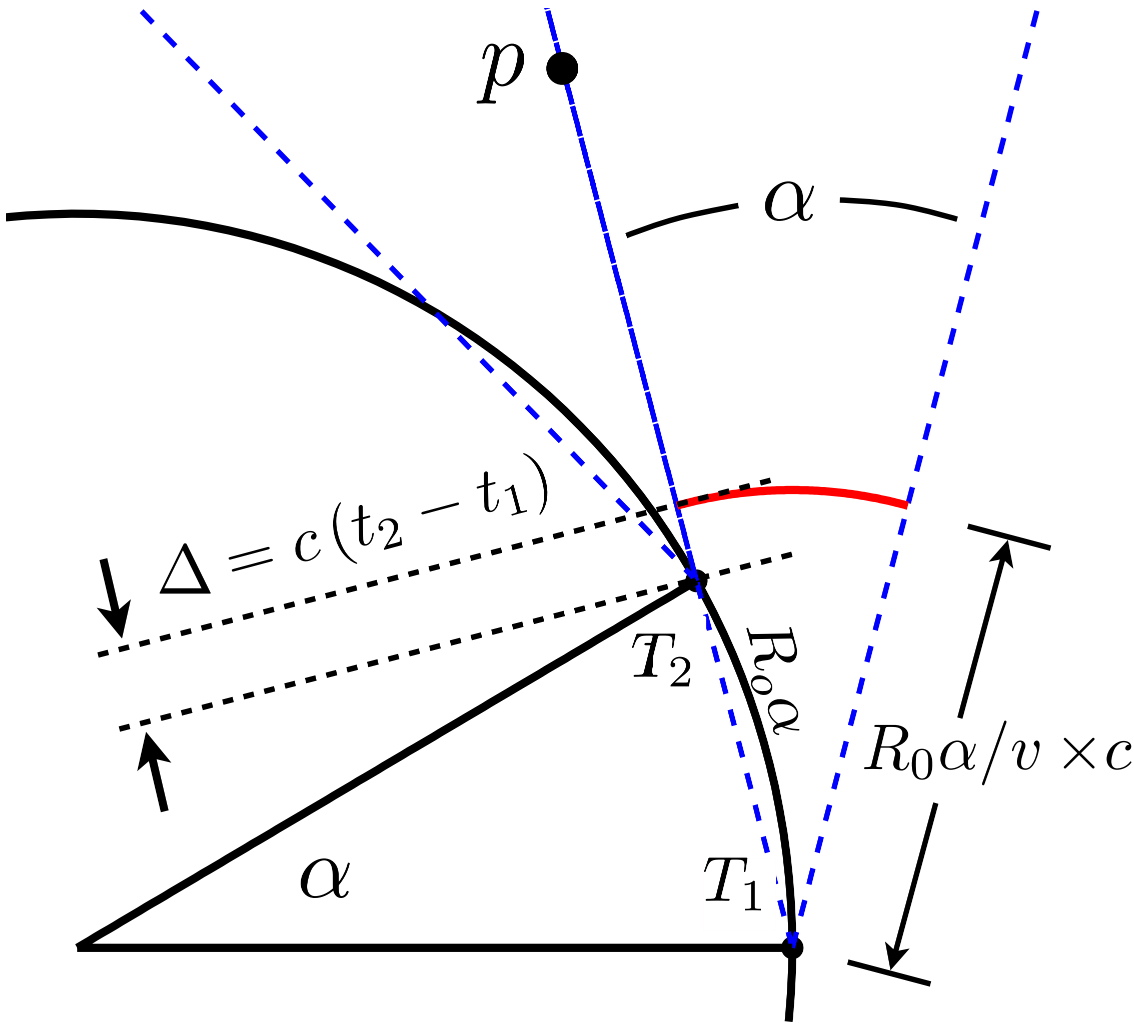
Time at source where the spotlight stops pointing at you

③ Then the kinematics of the emission process says that if the radiation is formed over time ΔT then it is observed to have time scale Δt

$$\Delta t = \frac{\Delta t}{\Delta T} \Delta T = (1 - n \cdot \beta) \frac{R_0}{\gamma V} \sim \frac{(1 + (\gamma\theta)^2)}{2\gamma^2} \frac{R_0}{\gamma c}$$

$$\Delta t \sim \frac{R_0 / c}{\gamma^3}$$

Figure Credit: Christina Athanasion et al, arXiv: 1001.3880



Estimate pg. 3

Can also see from geometry

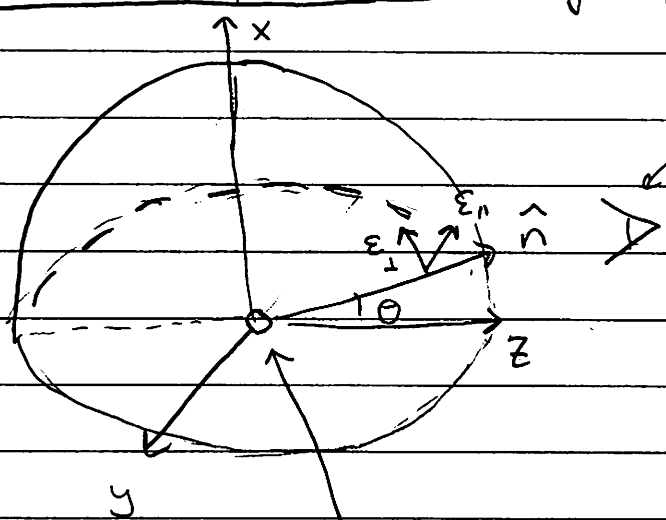
$$c\Delta t = \frac{R_0 \alpha c}{v} - R_0\alpha = R_0 \overbrace{\alpha}^{\sim \frac{1}{\gamma}} \overbrace{\left(\frac{1}{\beta} - 1\right)}^{1/\gamma^2}$$

$$\Delta t \sim \frac{R_0/c}{\gamma^3}$$

And

$$\Delta \omega \sim \frac{\gamma^3}{(R_0/c)}$$

The Sychrotron Spectrum Using Eq. (★★★)'



detector with incline θ relative to y, z plane

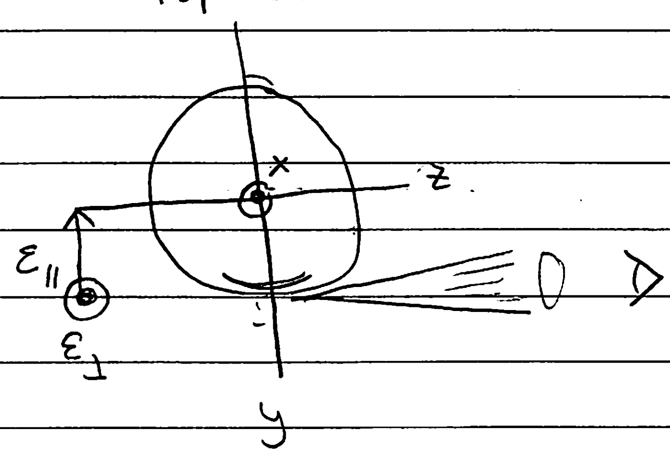
particle going around in a circle here in y, z plane

(1) Choose the observation direction, so that it lies in x - z plane:

$$\hat{n} = (\sin\theta, 0, \cos\theta)$$

(2) Take the particle going in the y - z plane:

Top View



$$r_*(T) = R_0(0, \cos\omega_0 T, \sin\omega_0 T)$$

$$\beta(T) = \frac{v_0}{c}(0, -\sin\omega_0 T, \cos\omega_0 T)$$

we are drawing the motion at T around T=0, and $\theta = 0$

$$\vec{\varepsilon}_{\perp} = (\cos\theta, 0, \sin\theta) \simeq \hat{x}, \quad \vec{\varepsilon}_{\parallel} = -\hat{y}$$

Using The Formula for $E(\omega)$, Eq (★★★)

Need $n \times n \times \beta = -\vec{\beta} + \vec{n}(n \cdot \vec{\beta})$

$$= \frac{V_0}{c}\left(\cos\omega_0 T \cos\theta \sin\theta,\ \sin\omega_0 T,\ -\cos\omega_0 T\,(1-\cos^2\theta)\right)$$

$$\simeq \frac{V_0}{c}\left(\theta,\ \omega_0 T,\ 0\right)$$

see definition of $\mathcal{E}_{\parallel}$ and $\mathcal{E}_{\perp}$.

$$= \theta\,\vec{\mathcal{E}}_{\perp} + -\left(\frac{cT}{R_0}\right)\vec{\mathcal{E}}_{\parallel}$$

Use $V_0 = \omega_0 R_0$.

Use $v \simeq c$

Then we approximate the phase

$$\phi = \omega\left(T - \frac{\vec{n}\cdot\vec{r}_*(T)}{c}\right) = \omega\left(T - \frac{R_0}{c}\sin\omega_0 T\cos\theta\right)$$

Expanding to cubic order with T, θ, small and $1-\beta \simeq 1/2\gamma^2 \simeq (\text{small})^2$ we have

$$\sin\omega_0 T \simeq \omega_0 T + \frac{1}{3}(\omega_0 T)^3, \quad \cos\theta \simeq 1 - \frac{\theta^2}{2}$$

use $V_0 = \omega_0 R_0$

So

$$\phi = \omega T\left(1 - \frac{V_0}{c}\cos\theta\right) + \frac{\omega}{3!}\frac{R_0\omega_0^3}{c}T^3$$

$$\simeq \frac{\omega}{2}\left[\left(\frac{1}{\gamma^2} + \theta^2\right)T + \frac{c^2}{3R^2}T^3\right]$$

use $\omega_0 \simeq \frac{c}{R}$

and use

$$\left(1 - \frac{V}{c}\cos\theta\right) \simeq \frac{1}{2\gamma^2} + \frac{\theta^2}{2}$$

So now we can evaluate $|E(\omega)|^2$, Eq (★★★)

From two pages back

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2}{16\pi^2} \frac{\omega^2}{c} \left| \int_{-\infty}^{\infty} dT \, e^{i\omega(T - \frac{n \cdot r_*(T)}{c})} \, \vec{n} \times \vec{n} \times \vec{\beta}(T) \right|^2$$

Find

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2}{16\pi^2} \frac{\omega^2}{c} \left| \overbrace{A_\perp \vec{\mathcal{E}}_\perp}^{\propto v_0} + \overbrace{(-A_\parallel)}^{\propto cT/R} \vec{\mathcal{E}}_\parallel \right|^2$$

Where

$$A_\perp = \theta \int_{-\infty}^{\infty} dT \, e^{\frac{i\omega}{2}\left[\left(\frac{1}{\gamma^2} + \theta^2\right)T + \frac{1}{3}\frac{c^2}{R^2}T^3\right]}$$

$$A_\parallel = \int_{-\infty}^{\infty} dT \, e^{\frac{i\omega}{2}\left[\left(\frac{1}{\gamma^2} + \theta^2\right)T + \frac{1}{3}\frac{c^2}{R^2}T^3\right]} \frac{cT}{R}$$

Now we rescale the retarded time and the frequency

$$x \equiv \frac{cT}{R_0} \frac{1}{(1/\gamma^2 + \theta^2)^{1/2}} \quad \text{and} \quad \xi = \frac{\omega R_0/c}{3\gamma^3} \left(1 + (\gamma\theta)^2\right)^{3/2}$$

This may seem mysterious but the rescalings are chosen so:

$$\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2} + \theta^2\right)T + \frac{1}{3}\frac{c^2 T^3}{R^2}\right] = \frac{3}{2}\xi\left(x + \frac{1}{3}x^3\right)$$

Evaluating Eq (★★★) pg. 4

Then

$$A_{\parallel} = \frac{R_0}{c}\overbrace{\left(\frac{1}{\gamma^2}+\theta^2\right)}^{\propto \xi^{2/3}} \underbrace{\int_{-\infty}^{\infty} dx\; e^{i\frac{3}{2}\xi\left(x+\frac{x^3}{3}\right)}\, x}_{\frac{2}{\sqrt{3}} K_{2/3}(\xi)}$$

← airy integral or modified bessel

$$A_{\perp} = \frac{R_0\theta}{c}\overbrace{\left(\frac{1}{\gamma^2}+\theta^2\right)^{\frac{1}{2}}}^{\propto \xi^{1/3}} \underbrace{\int_{-\infty}^{\infty} dx\; e^{i\frac{3}{2}\xi\left(x+\frac{1}{3}x^3\right)}}_{\frac{2}{\sqrt{3}} K_{1/3}(\xi)}$$

So then

$$2\pi \frac{dW}{d\omega\, d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left[\overbrace{\left(\frac{\omega R_0}{3c\gamma^3}\right)^{2/3}\left(\xi^{2/3}K_{2/3}(\xi)\right)^2}^{\text{parallel polarization contribution}} + \underbrace{\left(\frac{\omega R_0}{3c\gamma^3}\right)^{4/3}\left(\gamma\theta\,\xi^{1/3}K_{1/3}(\xi)\right)^2}_{\text{perpendicular polarization contribution}}\right]$$

We should Analyze this:

$$2\pi \frac{dW}{d\omega\, d\Omega} = \frac{q^2\gamma^2}{c}\, \underbrace{F\left(\frac{\omega\gamma^3}{(R_0/c)},\ \gamma\theta\right)}_{\text{dimensionless order 1 function}}$$

- The characteristic angle is set by $\gamma\theta \sim 1$ or

$$\theta \sim 1/\gamma \quad \leftarrow \text{ this is what we found previously}$$

- The characteristic frequency is set by $\omega\gamma^3/(R_o/c) \sim 1$ or

$$\omega \sim \frac{\gamma^3}{R_o/c}$$

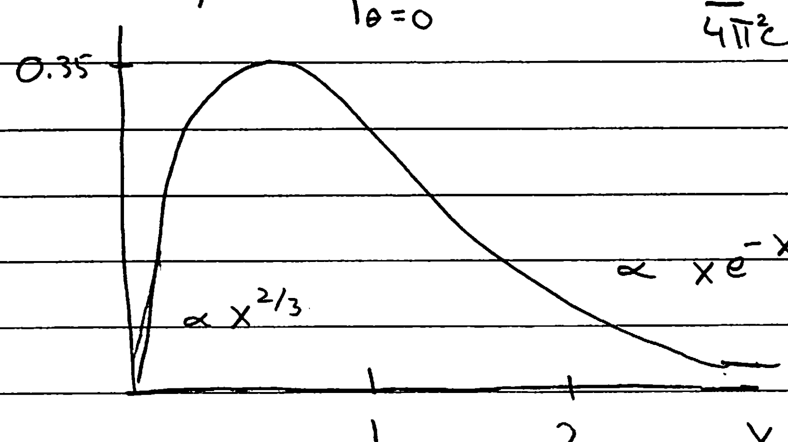
Analysis of Formula

- First determine the inplane $\theta = 0$ frequency spectrum.

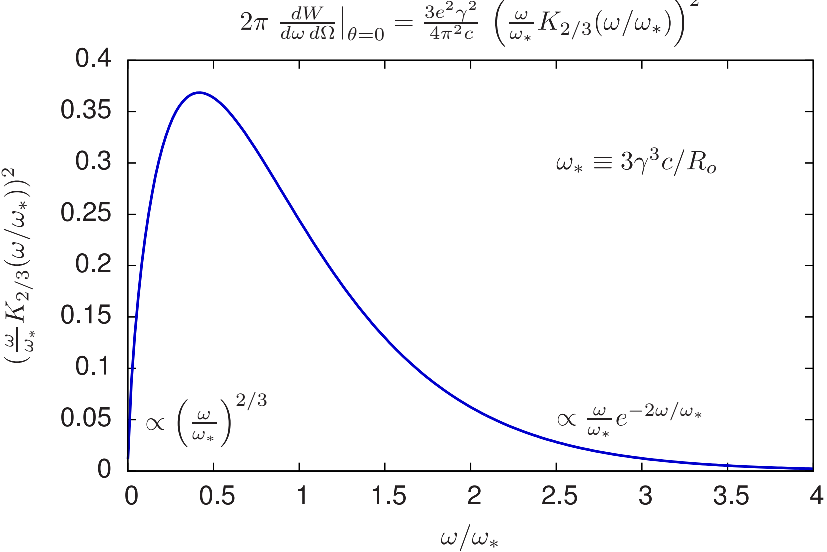
Then, $\xi = \frac{\omega R_o}{3c}\left(\frac{1}{\gamma^2} + \theta^2\right)^{3/2} \xrightarrow[\theta \to 0]{} \frac{\omega R_o}{3c\gamma^3} \equiv X$, so

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{3q^2}{4\pi^2 c}\gamma^2 \left(X K_{2/3}(X)\right)^2$$

So $2\pi\, dW/d\omega d\Omega \Big|_{\theta=0}$ in units $\frac{3e^2\gamma^2}{4\pi^2 c}$



$X \equiv \frac{1}{3}\frac{\omega R}{c\gamma^3}$ = frequency in units


$2\pi \left.\frac{dW}{d\omega\, d\Omega}\right|_{\theta=0} = \frac{3e^2\gamma^2}{4\pi^2 c} \left(\frac{\omega}{\omega_*} K_{2/3}(\omega/\omega_*)\right)^2$
$\omega_* \equiv 3\gamma^3 c/R_o$
$\propto \left(\frac{\omega}{\omega_*}\right)^{2/3}$
$\propto \frac{\omega}{\omega_*} e^{-2\omega/\omega_*}$
$\left(\frac{\omega}{\omega_*} K_{2/3}(\omega/\omega_*)\right)^2$
ω/ω_*
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.5
1
1.5
2
2.5
3
3.5
4