Relation Between Scattering Amplitude and Currents

The radiated field

$$A_{rad} = \frac{1}{4\pi r} \int d^3r_0 \; J(T, r_0)$$

For sinusoidal currents $J(t) = J_\omega e^{-i\omega t}$

$$T = t - \frac{r}{c} + \frac{n \cdot r_0}{c}$$

$$\vec{A}_{rad} = \frac{1}{4\pi r} e^{-i\omega(t - r/c)} \int d^3r_0 \; \frac{\vec{J}_\omega}{c}(r_0) \; e^{-i\omega \frac{n \cdot r_0}{c}}$$

$$= \frac{1}{4\pi r} e^{-i\omega t + ikr} \int d^3r_0 \; \frac{\vec{J}_\omega(r_0)}{c} \; e^{-i\vec{k} \cdot r_0}$$

Now

$$E_{rad} = n \times n \times \frac{1}{c} \frac{\partial A_{rad}}{\partial t}$$

$$= \frac{-i\omega}{4\pi r c} e^{-i\omega t + ikr} \; n \times n \times \int d^3r_0 \; \frac{\vec{J}_\omega}{c}(r_0) \; e^{-i\vec{k} \cdot \vec{r}_0}$$

Comparison gives $\quad E_{rad} = E_0 \frac{e^{ikr-i\omega t}}{r} \vec{f}(k)$

$$\vec{f}(k) = \frac{-ik}{4\pi E_0} \, n \times n \times \int d^3r_0 \, \frac{J_\omega(r_0)}{c} e^{-ik\cdot r}$$

And thus using $|n \times n \times V|^2 = |n \times V|^2$ we have ☆

$$\boxed{\frac{d\sigma}{d\Omega} = |\vec{f}(k)|^2 = \frac{k^2}{16\pi^2 E_0^2} \left| n \times \int d^3r_0 \, \frac{J_\omega(r_0)}{c} e^{-i\vec{k}\cdot\vec{r}_0} \right|^2}$$ ☆

This explicitly shows how the induced currents determine the cross section

Born Approximation

- To procede further we need to specify the currents. For dielectric media $J(t) = \partial_t P = \chi_e \partial_t E$

$$\vec{J}_\omega(r) = -i\omega \, \chi(\omega, r) \, E_\omega(r)$$

- Then in a weak field approximation we can consider the current to arise solely from the incoming light.

$$j_\omega(r) = -i\omega\chi(\omega) \left(E_\omega^{inc}(r) + E_\omega^{scatt}(r) \right)$$

$$\simeq -i\omega \, \chi(\omega) \, E_\omega^{inc}(r)$$

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Now define $\vec{k}_0 \equiv k\hat{z}$ ← incoming wave vector

$$E_{inc}(t) = \underbrace{\left[E_0 \vec{\mathcal{E}}_0 e^{i\vec{k}_0\cdot\vec{r}_0}\right]}_{E_\omega^{inc}(r)} e^{-i\omega t} \qquad e^{i\vec{k}_0\cdot\vec{r}_0} = e^{ikz_0}$$

So $\quad j_\omega(r_0) = -i\omega\,\chi(\omega, r_0)\, E_0 \vec{\mathcal{E}}_0 e^{i\vec{k}_0\cdot\vec{r}_0}$

And plugging into Eq ★★ on the previous page:

$$\frac{d\sigma}{d\Omega} = \frac{k^2}{16\pi^2 E_0^2}\left| n\times\int_{r_0} -\frac{i\omega}{c}\,\chi(\omega,r_0)\, E_0\vec{\mathcal{E}}_0 e^{i\vec{k}_0\cdot\vec{r}_0}\, e^{-i\vec{k}\cdot\vec{r}_0}\right|^2$$

And

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{k^2}{4\pi}\right)^2 \left|\vec{n}\times\vec{\mathcal{E}}_0\right|^2 \left|\int d^3r_0\, \chi(\omega,r_0)\, e^{-i(\vec{k}-\vec{k}_0)\cdot\vec{r}_0}\right|^2}$$

Example ①

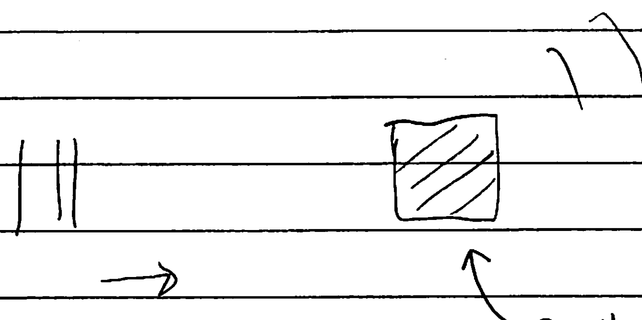
Two Examples @ Born Approximation — Dipole Limit

① Long wavelength limit $k \cdot L \ll 1$ then you can neglect the phase, finding

$$\int d^3r_0 \, \chi(\omega, r) \cdot 1 = \chi(\omega) V \equiv \alpha_E$$

The total dipole moment is

$$\vec{p} = \underbrace{\chi V}_{\alpha_E} \vec{E}$$



E_{inc}

scattering obj @ const polarizability and volume V

Thus in this limit:

$$\frac{d\sigma}{d\Omega} = \left(\frac{k^2}{4\pi}\right)^2 |n \times \mathcal{E}_0|^2 \alpha_E^2$$

$$= \frac{\alpha_E^2}{16\pi^2} \left(\frac{\omega}{c}\right)^4 \left(1 - |n \cdot \mathcal{E}_0|^2\right)$$

This is the same dipole scattering we discussed in the beginning.

For the dielectric sphere:

$$\alpha_E = 4\pi \left(\frac{\varepsilon - 1}{\varepsilon + 2}\right) a^3$$

$$\simeq \underbrace{(4\pi a^3)}_{V} \underbrace{(\varepsilon - 1)}_{\chi}$$

Born Approx Ex. 2

Example 2

(2) For a solid sphere of radius R the cross section is proportional to

$$\chi(\omega, \vec{q}) = \int_{\text{sphere}} d^3r \; \chi(\omega, r)\, e^{i\vec{q}\cdot\vec{r}} \qquad \vec{q} \equiv \vec{k} - \vec{k}_0$$

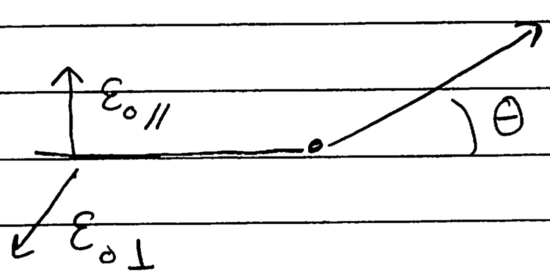
$$= 2\pi\, \chi(\omega) \int_0^R r^2 dr \int_{-1}^{1} d(\cos\theta)\; e^{iqr\cos\theta}$$

$$= 2\pi\, \chi(\omega) \int_0^R r^2 dr \left(\frac{\sin qr}{qr}\right) \qquad j_0(qr) \equiv \frac{\sin qr}{qr}$$

$$= 4\pi R^3 \chi(\omega)\, \frac{j_1(qR)}{qR} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

Now then we have to work out

$|n \times \mathcal{E}_0|^2$ averaged over polarizations of incoming light



Then using $|n \times \varepsilon_0|^2 = (1 - |n \cdot \varepsilon_0|^2)$
we have

$$|n \times \varepsilon_{0\parallel}|^2 = (1 - \sin^2\theta) = \cos^2\theta$$

$$|n \times \varepsilon_{0\perp}|^2 = (1 - 0) = 1$$

So

$$\text{ave } |\vec{n} \times \vec{\varepsilon}_0|^2 \text{ over pols} = \frac{1 + \cos^2\theta}{2}$$

And finally we need: $\vec{k}_0 = k\hat{z}$

$$q = |\vec{q}| = \sqrt{|\vec{k} - \vec{k}_0|^2} = (\vec{k}^2 - 2\vec{k}\cdot\vec{k}_0 + \vec{k}_0^2)^{1/2}$$

$$= \left[2k^2(1 - \cos\theta)\right]^{1/2}$$

$$= (4k^2 \sin^2\theta/2)^{1/2} = 2k\sin\theta/2$$

So we find

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{scat} \simeq \frac{R^2}{4} (k_0 R)^2 \chi^2 \left(\frac{1+\cos^2\theta}{2}\right) \frac{j_1^2(2kR\sin\theta/2)}{\sin^2\theta/2}}$$

The unpolarized cross section for for a sphere of radius R scattering light of wave number k is

$$\frac{d\sigma}{d\Omega} \simeq \frac{R^2}{4}(kR)^2\chi^2(\omega)\left[\frac{1+\cos^2\theta}{2}\left(\frac{j_1(2kR\sin\theta/2)}{\sin\theta/2}\right)^2\right] \tag{1}$$

where

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \tag{2}$$

is the sphereical bessel function. The term in square brackets is plotted below.