D Separation of Variables

D.1 Cartesian coordinates

(a) Laplacian

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi = 0 \]  \hspace{1cm} (D.1)

(b) Eigen functions along boundary vanishing at \( x = 0 \) and \( x = a \) and \( y = 0 \) and \( y = b \)

\[ \psi_{nm}(x, y) = \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi y}{b} \right) \quad n = 1 \ldots \infty \quad m = 1 \ldots \infty \]

(c) Orthogonality

\[ \int_0^a dx \int_0^b dy \psi_{nm} \psi_{n'm'} = \left( \frac{a}{2} \right) \left( \frac{b}{2} \right) \delta_{nn'} \delta_{mm'} \]

(d) Solution

\[ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ A_{nm} e^{-\gamma_{nm} z} + B_{nm} e^{+\gamma_{nm} z} \right] \psi_{nm}(x, y) \]  \hspace{1cm} (D.2)

where \( \gamma_{nm} = \sqrt{(n\pi/a)^2 + (m\pi/b)^2} \)
D.2 Spherical coordinates

(a) Laplacian
\[
\begin{bmatrix}
\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\end{bmatrix} \Phi = 0
\] (D.3)

(b) Eigen functions along boundary \( \theta, \phi \), regular at \( \theta = 0 \) and \( \pi \), \( 2\pi \) periodic in \( \phi \)
\[
\psi_{\ell m}(\theta, \phi) = Y_{\ell m}(\theta, \phi) \quad \ell = 0 \ldots \infty \quad m = -\ell \ldots \ell
\]

(c) Orthogonality:
\[
\int d\Omega Y^*_{\ell m}(\theta, \phi) Y_{\ell' m'}(\theta, \phi) = \delta_{\ell \ell'} \delta_{m m'}
\]

(d) Solution
\[
\Phi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[ A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right] Y_{\ell m}
\] (D.4)

(e) When there is no azimuthal dependence things simplify to
\[
\Phi = \sum_{\ell=0}^{\infty} \left[ A_{\ell} r^\ell + \frac{B_{\ell}}{r^{\ell+1}} \right] P_{\ell}(\cos \theta)
\] (D.5)

where \( P_{\ell}(\cos \theta) \) is the legendre polynomial, which up to a normalization if \( Y_{\ell 0}(\theta, \phi) \), satisfying the orthogonality
\[
\int_{-1}^{1} d(\cos \theta) P_{\ell}(\cos \theta) P_{\ell'}(\cos \theta) = \frac{2}{2\ell + 1} \delta_{\ell \ell'}
\]
D.3 Cylindrical Boundary: $z, \phi$ are the boundary.

(a) Laplacian:
\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \Phi = 0
\]  
(D.6)

(b) Eigenfunctions along boundary $z, \phi$ vanishing at $z = 0$ and $z = L$ and $2\pi$ periodic in $\phi$

\[
\psi_{nm}(z, \phi) = \sin(k_n z) e^{im\phi} \quad k_n \equiv \frac{n\pi}{L} \quad n = 1 \ldots \infty \quad m = -\infty \ldots \infty
\]

(c) Orthogonality:
\[
\int_0^L dz \int_0^{2\pi} \psi_{nm}(z, \phi) \psi_{nm}(z, \phi) = \frac{L}{2} (2\pi) \delta_{nn'} \delta_{mm'}
\]

(d) Solution:
\[
\Phi = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} [A_{nm} I_m(k_n \rho) + B_{nm} K_m(k_n \rho)] \psi_{nm}(z, \phi) \quad (D.7)
\]

Here $I_m(x)$ and $K_m(x)$ is the modified bessel function of the first and second kinds. Note that $K_{-m}(x) = K_m(x)$ and $I_{-m}(x)$
D.4 2D cylindrical coordinates

Boundary:
\[ \rho = \text{const} \]
\[ \phi \text{ changing} \]

(a) Laplacian:
\[
\left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right] \Phi = 0 \quad (D.8)
\]

(b) Eigenfunctions along boundary \( \phi \): 2\( \pi \) periodic in \( \phi \)
\[ \psi_m(\phi) = e^{im\phi} \quad m = -\infty \ldots \infty \]

(c) Orthogonality
\[
\int_0^{2\pi} \psi_m^*(\phi) \psi_m'(\phi) = 2\pi \delta_{mm'} \quad (D.9)
\]

(d) Solution
\[
\Phi = A_0 + B_0 \ln \rho + \sum_{m=-\infty}^{\infty} \left( A_m \rho^{|m|} + \frac{B_m}{\rho^{-|m|}} \right) \psi_m
\]
D.5 Cylindrical Boundary: $\rho, \phi$ are the boundary

**Boundary specified on $\rho, \phi$ surface**

- $\varphi = 0$ on sides
- $z = 0$
- $z = L$
- $\varphi(\rho, \phi, z = L) = \varphi_o(\rho, \phi)$

(a) Laplacian:

$$\left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] \Phi = 0 \quad (D.10)$$

(b) Eigenfunctions along boundary $\rho, \phi$ vanishing at $\rho = R$ and regular at $\rho = 0$, $2\pi$ periodic in $\phi$:

$$\psi_{mn}(\rho, \phi) = J_m(k_{mn} \rho) e^{im\phi} \quad n = 1 \ldots \infty \quad m = -\infty \ldots \infty$$

Here:

$$k_{mn} = \frac{x_{mn}}{R} \quad (D.11)$$

where $x_{mn}$ is the $n$-th zero of the $m$-th Bessel function, *e.g.* the zeros of $J_0(x)$ are

$$(x_{01}, x_{02}, x_{03}) = 2.40483, 5.52008, 8.65373 \quad (D.12)$$

These are given by $x_{mn} = \text{BesselZero}[m, n]$ in Mathematica. Note also that $J_{-m}(x) = J_m(x)$

(c) Orthogonality:

$$\int_0^R \rho d\rho \int_0^{2\pi} \psi_{mn}(\rho, \phi) \psi_{mn}(\rho, \phi) = \left( \frac{R^2}{2} \left[ J_{m+1}(k_{mn} R) \right]^2 \right) (2\pi) \delta_{nn'} \delta_{mm'}$$

(d) Solution:

$$\Phi = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \left[ A_{mn} e^{-k_{mn} z} + B_{mn} e^{k_{mn} z} \right] \psi_{mn}(\rho, \phi) \quad (D.13)$$
D.6 Continuum Forms and Fourier and Hankel Transforms

In each case we are expanding two directions of the solution in a complete set of eigenfunctions

\[
\langle x| F \rangle = \frac{1}{C_n} \sum_n \langle x| n \rangle \langle n| F \rangle ,
\]  
(D.14)

and solving the laplace equation to find the dependence on the third direction.

(a) For the cartesian case when \( a \) and \( b \) go to infinity. The sum becomes an integral and the sum over \( n \) and \( m \) becomes a 2D fourier transform

\[
\Phi = \int \frac{d^2 k}{(2\pi)^2} e^{ik_\perp \cdot x_{\perp}} \left[ A(k_\perp) e^{-k_{\perp} z} + B(k_\perp) e^{k_{\perp} z} \right].
\]

We are using the fact that any function in the \( x, y \) plane (in particular the boundary condition \( \Phi_o(x, y) \)) can be expressed as a fourier transform pairs

\[
F(x, y) \equiv \int \frac{d^2 k}{(2\pi)^2} \left[ e^{ik_\perp \cdot x_{\perp}} \right] F(k_x, k_y),
\]

(D.15)

\[
F(k_x, k_y) \equiv \int d^2 x_{\perp} \left[ e^{-ik_\perp \cdot x_{\perp}} \right] F(x, y).
\]

(D.16)

(b) For the cylindrical case when \( L \) goes to \( \infty \), the sum over \( n \) becomes an integral yielding

\[
\Phi = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[ e^{ikz} e^{im\phi} \right] \left[ A(k) I_m(|k|\rho) + B(k) K_m(|k|\rho) \right]
\]

We are using the fact that any regular function of \( z \) and \( \phi \) (in particular the boundary condition \( \Phi_o(z, \phi) \)) can be written in terms of its fourier components

\[
F(z, \phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[ e^{ikz} e^{im\phi} \right] F_m(k),
\]

(D.17)

\[
F_m(k) = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \left[ e^{-ikz} e^{-im\phi} \right] F(z, \phi)
\]

(D.18)

(c) Finally for the second cylindrical case when the radius goes to infinity

\[
\Phi = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk \left[ J_m(k\rho) e^{im\phi} \right] \left[ A(k) e^{-kz} + B(k) e^{kz} \right]
\]

(D.19)

We are using the fact that any regular cylindrical function of \( \rho \) and \( \phi \) (in particular the boundary condition \( \Phi_o(\rho, \phi) \)) can be written as Hankel transform

\[
F(\rho, \phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk \left[ J_m(k\rho) e^{im\phi} \right] F_m(k)
\]

(D.20)

\[
F_m(k) = \int_0^{2\pi} d\phi \int_0^{\infty} d\rho \rho \left[ J_m(k\rho) e^{-im\phi} \right] F(\rho, \phi)
\]

(D.21)