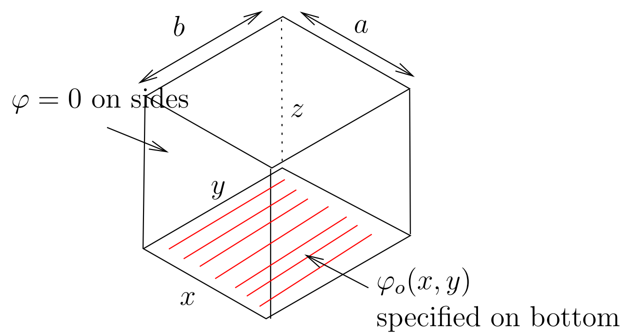
D Separation of Variables

D.1 Cartesian coordinates



(a) Laplacian

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Phi = 0 \tag{D.1}$$

(b) Eigen fucntions along boundary vanishing at $x = 0$ and $x = a$ and $y = 0$ and $y = b$

$$\psi_{nm}(x, y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \qquad n = 1\ldots\infty \qquad m = 1\ldots\infty$$

(c) Orthogonality

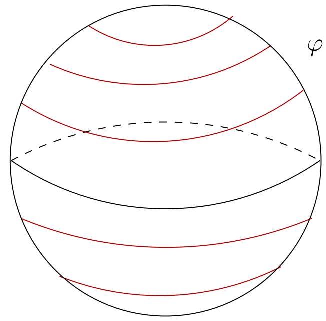
$$\int_0^a dx \int_0^b dy\, \psi_{nm}\, \psi_{n'm'} = \left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\delta_{nn'}\delta_{mm'}$$

(d) Solution

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{nm}e^{-\gamma_{nm}z} + B_{nm}e^{+\gamma_{nm}z}\right]\psi_{nm}(x, y) \tag{D.2}$$

where $\gamma_{nm} = \sqrt{(n\pi/a)^2 + (m\pi/b)^2}$

D.2 Spherical coordinates



(a) Laplacian

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\Phi = 0 \tag{D.3}$$

(b) Eigen fucntions along boundary θ, ϕ, regular at $\theta = 0$ and π, 2π periodic in ϕ

$$\psi_{\ell m}(\theta,\phi) = Y_{\ell m}(\theta,\phi) \qquad \ell = 0\ldots\infty \quad m = -\ell\ldots\ell$$

(c) Orthogonality:

$$\int d\Omega\, Y^*_{\ell m}(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$$

(d) Solution

$$\Phi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[A_{\ell m}r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}\right]Y_{\ell m} \tag{D.4}$$

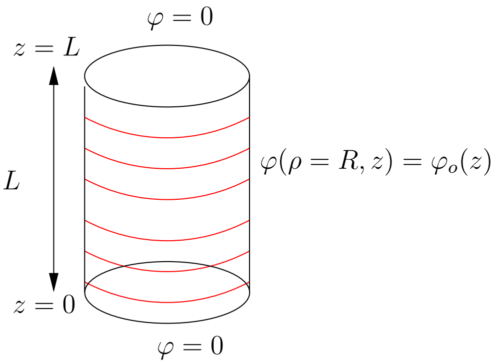
(e) When there is no azimuthal dependence things simplify to

$$\Phi = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right]P_\ell(\cos\theta) \tag{D.5}$$

where $P_\ell(\cos\theta)$ is the legendre polynomial, which up to a normalization if $Y_{\ell 0}(\theta,\phi)$, satisfying the orthogonality

$$\int_{-1}^{1} d(\cos\theta)P_\ell(\cos\theta)P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

D.3 Cylindrical Boundary: z, ϕ are the boundary.



(a) Laplacian:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]\Phi = 0 \tag{D.6}$$

(b) Eigenfunctions along boundary z, ϕ vanishing at $z = 0$ and $z = L$ and 2π periodic in ϕ

$$\psi_{nm}(z,\phi) = \sin(k_n z)\, e^{im\phi} \qquad k_n \equiv \frac{n\pi}{L} \qquad n = 1\ldots\infty \quad m = -\infty\ldots\infty$$

(c) Orthogonality:

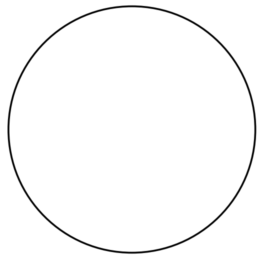
$$\int_0^L dz \int_0^{2\pi} \psi_{nm}(z,\phi)\,\psi_{nm}(z,\phi) = \frac{L}{2}\,(2\pi)\delta_{nn'}\delta_{mm'}$$

(d) Solution:

$$\Phi = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left[A_{nm}I_m(k_n\rho) + B_{nm}K_m(k_n\rho)\right]\psi_{nm}(z,\phi) \tag{D.7}$$

Here $I_\nu(x)$ and $K_\nu(x)$ is the modified bessel function of the first and second kinds. Note that $K_{-m}(x) = K_m(x)$ and $I_{-m}(x)$

D.4 2D cylindrical coordinates



(a) Laplacian:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]\Phi = 0 \tag{D.8}$$

(b) Eigenfunctions along boundary ϕ: 2π periodic in ϕ

$$\psi_m(\phi) = e^{im\phi} \qquad m = -\infty \ldots \infty$$

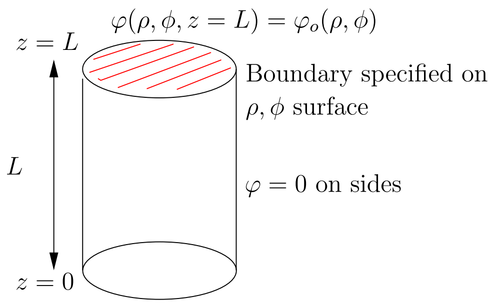
(c) Orthogonality

$$\int_0^{2\pi} \psi_m^*(\phi)\psi_{m'}(\phi) = 2\pi\delta_{mm'} \tag{D.9}$$

(d) Solution

$$\Phi = A_0 + B_0 \ln\rho + \sum_{m=-\infty}^{\infty}\left(A_m\rho^{|m|} + \frac{B_m}{\rho^{-|m|}}\right)\psi_m$$

D.5 Cylindrical Boundary: ρ, ϕ are the boundary



(a) Laplacian:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right]\Phi = 0 \tag{D.10}$$

(b) Eigenfunctions along boundary ρ, ϕ vanishing at $\rho = R$ and regular at $\rho = 0$, 2π periodic in ϕ:

$$\psi_{mn}(\rho, \phi) = J_m\left(k_{mn}\rho\right) e^{im\phi} \qquad n = 1 \dots \infty \;\; m = -\infty \dots \infty$$

Here:

$$k_{mn} = \frac{x_{mn}}{R} \tag{D.11}$$

where x_{mn} is the n-th zero of the m-th Bessel function, *e.g.* the zeros of $J_0(x)$ are

$$(x_{01}, x_{02}, x_{03}) = 2.40483,\ 5.52008,\ 8.65373 \tag{D.12}$$

These are given by $x_{mn} = \mathrm{BesselZeroJ[m, n]}$ in Mathematica. Note also that $J_{-m}(x) = J_m(x)$

(c) Orthogonality:

$$\int_0^R \rho d\rho \int_0^{2\pi} \psi_{mn}(\rho, \phi)\, \psi_{mn}(\rho, \phi) = \left(\frac{R^2}{2}\left[J_{m+1}(k_{mn}R)\right]^2\right)(2\pi)\,\delta_{nn'}\delta_{mm'}$$

(d) Solution:

$$\Phi = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left[A_{mn}e^{-k_{mn}z} + B_{nm}e^{k_{mn}z}\right]\psi_{mn}(\rho, \phi) \tag{D.13}$$

D.6 Continuum Forms and Fourier and Hankel Transforms

In each case we are expanding two directions of the solution in a complete set of eigenfunctions

$$\langle x|F\rangle = \frac{1}{C_n}\sum_n \langle x|n\rangle\,\langle n|F\rangle\ , \tag{D.14}$$

and solving the laplace equation to find the dependence on the third direction.

(a) For the cartesian case when a and b go to infinity. The sum becomes an integral and the sum over n and m becomes a 2D fourier transform

$$\Phi = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}\left[A(\boldsymbol{k}_\perp)e^{-k_\perp z} + B(\boldsymbol{k}_\perp)e^{k_\perp z}\right]\ .$$

We are using the fact that any function in the x, y plane (in particular the boundary condition $\Phi_o(x, y)$) can be expressed as a fourier transform pairs

$$F(x,y) \equiv \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2}\ \left[e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}\right]\ F(k_x,k_y)\,, \tag{D.15}$$

$$F(k_x,k_y) \equiv \int d^2\boldsymbol{x}_\perp\ \left[e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}\right]\ F(x,y)\,. \tag{D.16}$$

(b) For the cylindrical case when L goes to ∞, the sum over n becomes an integral yielding

$$\Phi = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\kappa}{2\pi}\ \left[e^{i\kappa z}e^{im\phi}\right]\ \left[A(\kappa)I_m(|\kappa|\rho) + B(k)K_m(|\kappa|\rho)\right]$$

We are using the fact that any regular function of z and ϕ (in particular the boundary condition $\Phi_o(z, \phi)$) can be written in terms of its fourier components

$$F(z,\phi) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\kappa}{2\pi}\ \left[e^{i\kappa z}e^{im\phi}\right]\ F_m(\kappa) \tag{D.17}$$

$$F_m(\kappa) = \int_0^{2\pi} d\phi\int_{-\infty}^{\infty} dz\ \left[e^{-i\kappa z}e^{-im\phi}\right]\ F(z,\phi) \tag{D.18}$$

(c) Finally for the second cylindrical case when the radius goes to infinity

$$\Phi = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} kdk\ \left[J_m(k\rho)e^{im\phi}\right]\left[A(k)e^{-kz} + B(k)e^{kz}\right] \tag{D.19}$$

We are using the fact that any regular cylindrical function of ρ and ϕ (in particular the boundary condition $\Phi_o(\rho, \phi)$) can be written as *Hankel* transform

$$F(\rho,\phi) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} kdk\ \left[J_m(k\rho)e^{im\phi}\right]\ F_m(k) \tag{D.20}$$

$$F_m(k) = \int_0^{2\pi} d\phi\int_0^{\infty}\rho d\rho\ \left[J_m(k\rho)e^{-im\phi}\right]\ F(\rho,\phi) \tag{D.21}$$