Vectors and Tensors

- We will use a new notation for vectors, which is very common,

$$\vec{V} = V^1 \vec{e}_1 + V^2 \vec{e}_2 + V^3 \vec{e}_3 = \sum_{i=1}^{3} V^i \vec{e}_i$$

where $(V^x, V^y, V^z) = (V^1, V^2, V^3)$ and $(\hat{\imath}, \hat{\jmath}, \hat{k}) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$

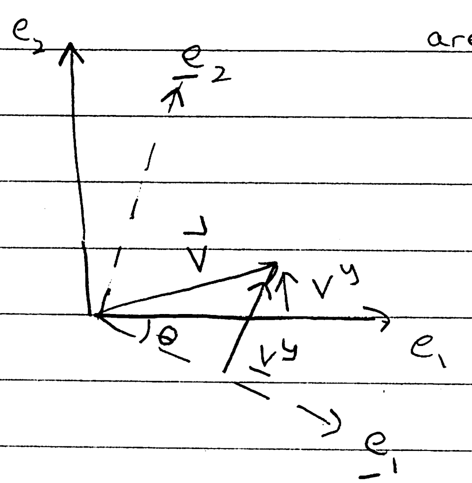
Then we use a summation convention, where repeated indices are summed from 1 to 3.

$$\vec{V} = V^i \vec{e}_i \qquad (\text{same as } \vec{V} = \sum V^i \vec{e}_i)$$

- Vectors are physical objects:

If the coordinates are rotated $\vec{V}$ remains unchanged. But the components V^i are changed (see figure which shows how V^y is changed), and the basis vectors $\vec{e}_i$ are also changed (rotated)

(Fig ★)



$$\underline{V}^i = R^i{}_j \, V^j \leftarrow \text{original vector components}$$

$\uparrow$ rotated vector components $\qquad$ $\uparrow$ rotation matrix

We use the summation convention here.

Think of v^i as a column: $\begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix} \equiv \begin{pmatrix} v^i \\ | \\ | \end{pmatrix}$

Then the rotated vector is

$$\begin{pmatrix} \underline{v}^i \\ | \\ | \end{pmatrix} = \begin{array}{c|c} & j \rightarrow \\ \hline i \downarrow & \begin{pmatrix} R^i{}_j \end{pmatrix} \end{array} \begin{pmatrix} v^j \\ | \\ | \end{pmatrix}$$

where, for the rotation shown in Fig ✩, R is:

$$(R^i{}_j) = \begin{array}{c|c} & j \rightarrow \\ \hline i \downarrow & \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \end{array}$$

Note: Rotations don't change norm. So

$$\underline{v}^T \underline{v} = v^T \underbrace{R^T R}_{\mathbb{1}} v = v^T v$$

So: R is orthogonal, $R^{-1} = R^T$

• Then, since $\vec{v}$ is unchanged under rotation, we need that the basis vectors transform as the inverse transformation and as a row:

$$\underline{\vec{e}}_j = \vec{e}_i \,(R^{-1})^i{}_j$$

rotated basis vector　　　　original basis

Think of basis vectors as a row

$$(\underline{e}_1\ \underline{e}_2\ \underline{e}_3) = (e_1\ e_2\ e_3)\begin{pmatrix} R^{-1} \end{pmatrix}$$

In this way, $\vec{v}$ is unchanged under rotation:

$$\underline{\vec{v}} = \underline{\vec{e}}_i\, \underline{v}^i$$

$$= (e_i \ldots)\underbrace{\begin{pmatrix} R^{-1} \end{pmatrix}\begin{pmatrix} R \end{pmatrix}}_{\mathbb{1}}\begin{pmatrix} v^i \\ \vdots \end{pmatrix}$$

$$= \vec{e}_i\, v^i = \vec{v}$$

We used

$$(R^{-1})^i{}_j\, R^j{}_k = \delta^i{}_k \equiv \begin{cases} \text{identity} \\ \text{matrix} \end{cases} \equiv \begin{matrix} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{matrix}$$

- Contravariant / Covariant indices

For every set of upstairs indices (contravariant) (V^x, V^y, V^z) define the lowered (covariant) components (V_x, V_y, V_z) which transform as a row according to the inverse matrix

$$\boxed{\underline{V}_i = V_i (R^{-1})^i{}_j}$$

$$(\underline{V}_i \dots) = (V_i \dots) \begin{pmatrix} R^{-1} \end{pmatrix}$$

Now since $R^{-1} = R^T$ (since rotations preserve length i.e. $\underline{x}^T \underline{x} = x^T x$), we see that

$$V_i = V_i (R^T)^i{}_j = V_i (R)_j{}^i .$$

i.e

$$\underline{V}_i = (R)_j{}^i V_i$$

But this is the same transformation rule as for upstairs indices. So up and down are the same for rotations.

$$V_x = V^x \quad , \quad \text{or} \quad V_i = \delta_{ij} V^j \quad V^i = \delta^{ij} V_j$$

So indices are raised and lowered with δ^{ij}, + δ_{ij}

Similary define contravariant basis vectors

$$\vec{e}^{\,i} = \delta^{ij} \vec{e}_j$$

Which transform as a column vector

$$\underline{e}^{\,i} = R^i{}_j e^j$$

So that the vector is rotationally invariant

$$\vec{V} = V_i \vec{e}^{\,i} = V^i \vec{e}_i$$

Dot-Products and Cross-Products:

① $\vec{a}\cdot\vec{b} = (a^i \vec{e}_i)(b^j \vec{e}_j) = a^i b^j \underbrace{\vec{e}_i \cdot \vec{e}_j}_{}$ $= \delta_{ij}$

$= a^i b^j \delta_{ij} = a_i b^i$

Rotationally invariant. Prf. easy. Contracted indices are invariant.

② To define cross product, need the epsilon tensor:

$$\varepsilon^{ijk} = \begin{cases} \pm 1 & \text{for } i,j,k \text{ even/odd permutation of } 1,2,3 \\ 0 & \end{cases}$$

e.g $\varepsilon^{123} = \varepsilon^{312} = \varepsilon^{231} = \varepsilon^{123} = -\varepsilon^{213} = -\varepsilon^{321} = +1$

Then this is a tensor which is the same after rotation. Prf uses $\det R = 1$.

$$\det R = R^1{}_i R^2{}_j R^3{}_k \varepsilon^{ijk}$$

$$= \begin{vmatrix} R^1{}_i & \cdot & \cdot \\ R^2{}_j & \cdot & \cdot \\ R^3{}_k & \cdot & \cdot \end{vmatrix}$$

Cross Products pg. 2

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix} = e_i \, \varepsilon^{ijk} \, a_j b_k$$

So

$$(\vec{a}\times\vec{b})^i = \varepsilon^{ijk} a_j b_k = i\text{-th} \overset{\text{contravariant}}{\vee} \text{ component of } \vec{a}\times\vec{b}$$

The

$$(a\times(b\times c))^i = \varepsilon^{ijk} a_j \, \varepsilon^{klm} b_l c_m$$

$$= \underbrace{\varepsilon^{ijk}\varepsilon^{klm}} a_j b_l c_m$$

Think about it: for example, consider ε^{ij3}. then for ε^{ij3}, is non-zero for $(i,j) = (1,2)$ and $(i,j) = (2,1)$,

$$\varepsilon^{123} = -\varepsilon^{213} = 1$$

So thinking along these lines we conclude

$$\varepsilon^{ijk}\varepsilon^{klm} = \boxed{\varepsilon^{ijk}\varepsilon^{lmk} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}}$$

So

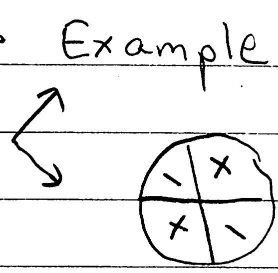
$$(a\times(b\times c))^i = (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})\, a_j b_l c_m$$

$$= b^i(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{b})\, c^i,$$ the "bac - abc" rule:

$$\boxed{\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\,(a\cdot c) - (\vec{a}\cdot\vec{b})\,\vec{c}}$$ ← very important.

Tensors

- Example: Want to describe the anisotropy of the charge distribution, and its orientation. Sort of described by two vectors. We will see that the right concept is the quadrupole tensor



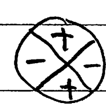
$$Q^{ij} = \int d^3x \, \rho(\vec{x}) \left(x^i x^j - \frac{1}{3} x^2 \delta^{ij} \right)$$

Rotations, rotate each arm of the tensor:

$$\underline{Q}^{ij} = R^i{}_l \, R^j{}_m \, Q^{lm}$$

original tensor

rotated tensor



Then, $\overleftrightarrow{Q} = Q^{ij} \vec{e}_i \vec{e}_j$, is a tensor and is unchanged under rotation of coordinates, since:

$$\underline{\vec{e}}_i \, \underline{\vec{e}}_j = \vec{e}_l \, \vec{e}_m \, (R^{-1})^l{}_i \, (R^{-1})^m{}_j$$

Derivative Operations:

$$\boxed{\partial_i \equiv \frac{\partial}{\partial x^i}}$$

grad $= (\vec{\nabla} S)_i = \partial_i S$

curl $= (\nabla \times \vec{V})^i = \varepsilon^{ijk} \partial_j V_k$

div $= \nabla \cdot \vec{V} = \partial_i V^i = \partial_x V^x + \partial_y V^y + \partial_z V^z$

laplacian $\nabla \cdot \vec{\nabla} S = \partial_i \partial^i$

The b(ac) − (ab)c rule plays an important role:

$$\nabla \times (\nabla \times \vec{C}) = \vec{\nabla}(\nabla \cdot \vec{V}) - \nabla^2 \vec{C}$$

- Homework: use the b(ac) − (ab)c rule to derive the wave equation

Helmholtz Theorems:

(1) If $\vec{\nabla}\cdot\vec{C} = 0$, then there exsists $\vec{D}$ such that:

$$\vec{C} = \vec{\nabla}\times\vec{D}.$$

(2) If $\nabla\times\vec{C} = 0$, then there exsists a scalar field S such that:

$$\vec{C} = -\vec{\nabla} S,$$

I wont prove it (but see homework) but I will show the converse, i.e.

(1) $\nabla\cdot(\vec{\nabla}\times\vec{D}) = 0$ and (2) $\vec{\nabla}\times(\vec{\nabla}S) = 0$

Prf.

(1) $$\partial_i C^i = \partial_i \underbrace{\varepsilon^{ijk}\partial_j D_k}_{(\nabla\times C)^i} = \varepsilon^{ijk}\partial_i\partial_j D_k = 0$$

Because $\varepsilon^{ijk} = -\varepsilon^{jki}$ is antisymmetric while $\partial_i\partial_j = \partial_j\partial_i$ is symmetric, $\partial_x\partial_y - \partial_y\partial_x = 0$.

(2) Similarly, we show $\nabla\times\vec{\nabla}S = 0$

$$\underbrace{\varepsilon^{ijk}\partial_j C_k}_{(\nabla\times C)^i} = \varepsilon^{ijk}\partial_j\partial_k S = 0$$

These are statements of differential forms $ddD = 0$

Maxwell Equations & The Helmholtz Theorems

The maxwell equations + Helmholtz theorems lead to two very important results:

I. Current Conservation
II. Gauge Potentials

First we write the MEqs. again

with source (currents)
$$\begin{cases} \nabla \cdot E = \rho \\ \nabla \times B = \vec{j}/c + 1/c\, \partial_t \vec{E} \end{cases}$$

without source
$$\begin{cases} \nabla \cdot B = 0 \\ -\nabla \times E = \frac{1}{c} \partial_t \vec{B} \end{cases}$$

I. Current Conservation.

Take the time derivative of the first equation, $\partial_t \nabla \cdot E = \partial_t \rho$, and the divergence (times c) of the second

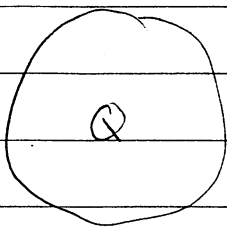
$$c\,\nabla \cdot (\nabla \times B) = 0 = \nabla \cdot \vec{j} + \nabla \cdot \partial_t \vec{E}$$

Adding these two results

$$\boxed{\partial_t \rho + \nabla \cdot \vec{j} = 0} \Leftarrow \text{conservation law}$$

Thus we see that maxwell equations are only consistent if charge is conserved. The conservation law imply's charge conservation

$$\partial_t Q = \int_{\text{Volume}} \partial_t \rho \, dV = \int_V -\nabla \cdot \vec{j} \, dV$$



$$= -\int_{\text{Surface}} \vec{j} \cdot d\vec{S} \longrightarrow 0$$ if the surface is taken far away

II. Gauge Potentials

Now consider the source-free equations.
(In the previous case we studied the sourced eqs.)

We can "trivially" solve these two eqs. using Helmholtz. From the third maxwell eqs

$$\nabla \cdot B = 0, \quad \text{so} \quad \boxed{B = \nabla \times \vec{A}}$$

where $\vec{A}$ is known as the vector potential.
Similarly. From the fourth Maxwell Equation

$$-\nabla \times E = \frac{1}{c} \partial_t \vec{B}$$

we find

$$-\nabla \times E = \frac{1}{c}\partial_t(\nabla \times \vec{A})$$

$$-\nabla \times (\vec{E} + \frac{1}{c}\partial_t \vec{A}) = 0$$

Thus we can write $\vec{E} + \frac{1}{c}\partial_t \vec{A}$ as a gradient

$$\vec{E} + \frac{1}{c}\partial_t \vec{A} = -\vec{\nabla}\phi \Leftarrow \text{the scalar potential (voltage)}$$

or

$$\boxed{\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\phi}$$

From basically now on we will solve for $(\phi, \vec{A})$ instead of $(\vec{E}, \vec{B})$, since working with these variables we automatically satisfy two of the four maxwell eqs