Functions of a Complex Variable

- A complex function takes in a complex number and spits out a complex number

$$z = x + iy = re^{i\theta}$$

$$f(x,y) = u(x,y) + iv(x,y)$$

Example 1

$$f(z) = z^2 = r^2 e^{i2\theta}$$

$$= (x+iy)(x+iy)$$

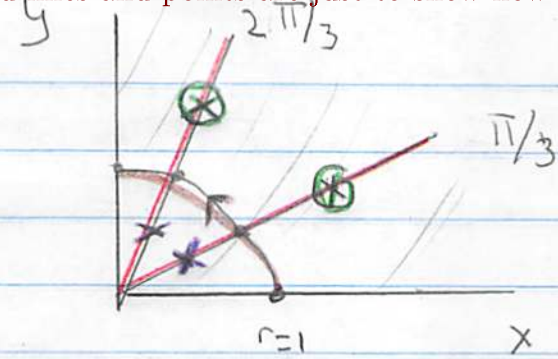
$$= \underbrace{(x^2 - y^2)}_{\equiv u} + \underbrace{2ixy}_{\equiv iv}$$

- You should think of complex functions as mappings from the complex plane to the complex plane

- You must specify the domain of the complex function

We are the map z^2, which maps points the x, y plane to the u, v plane, $(x, y) \mapsto (u, v)$

The colored lines and points are just to show how these objects are transformed by the map



domain = quadrant

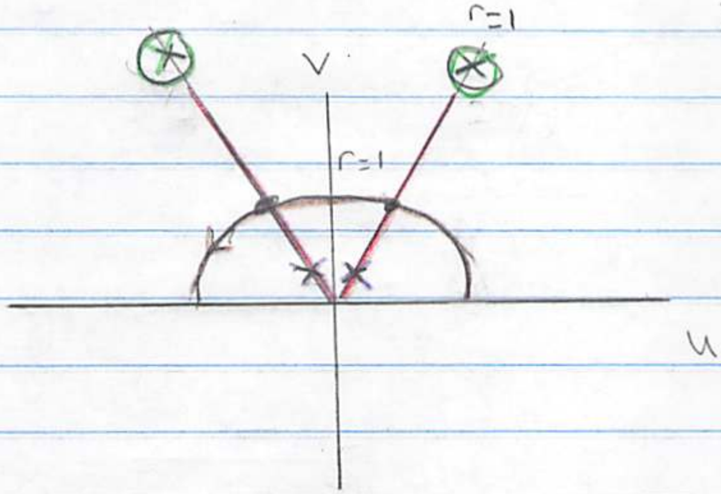
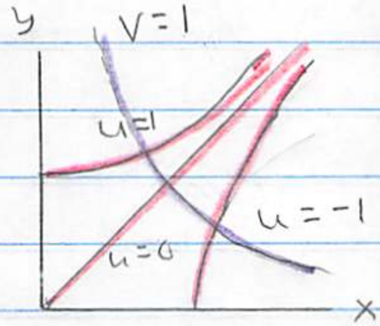


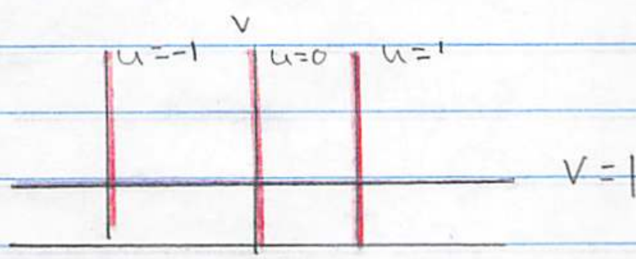
image = upper half plane

Similarly we can examine which lines of const u, v map to x, y



domain = quadrant

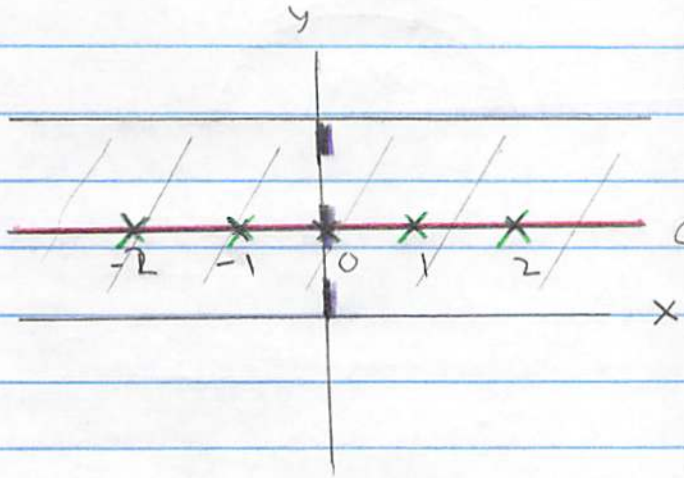
image = upper half plane



Example 2

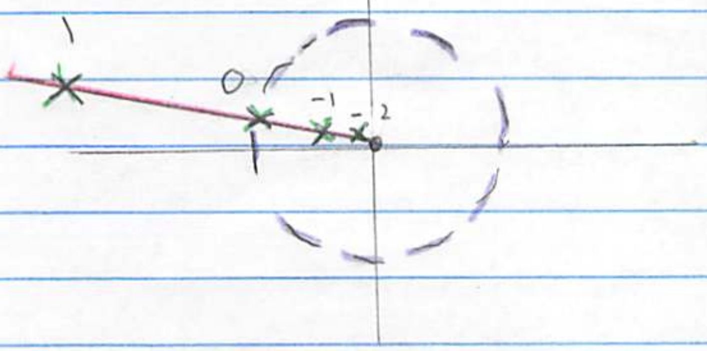
$$e^z = e^{x+iy}$$

$$= e^x (\cos y + i \sin y)$$



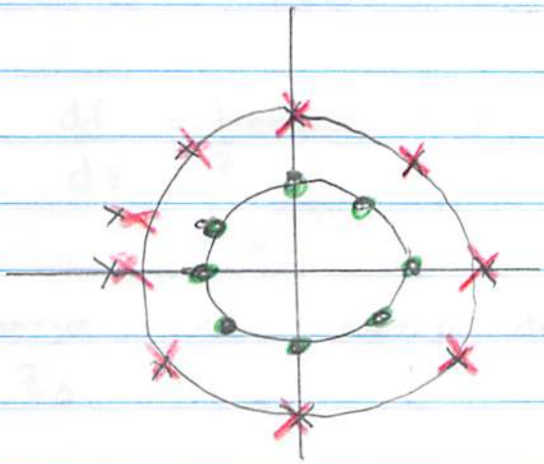
domain: Strip from $0 \ldots 2\pi i$

image: whole plane



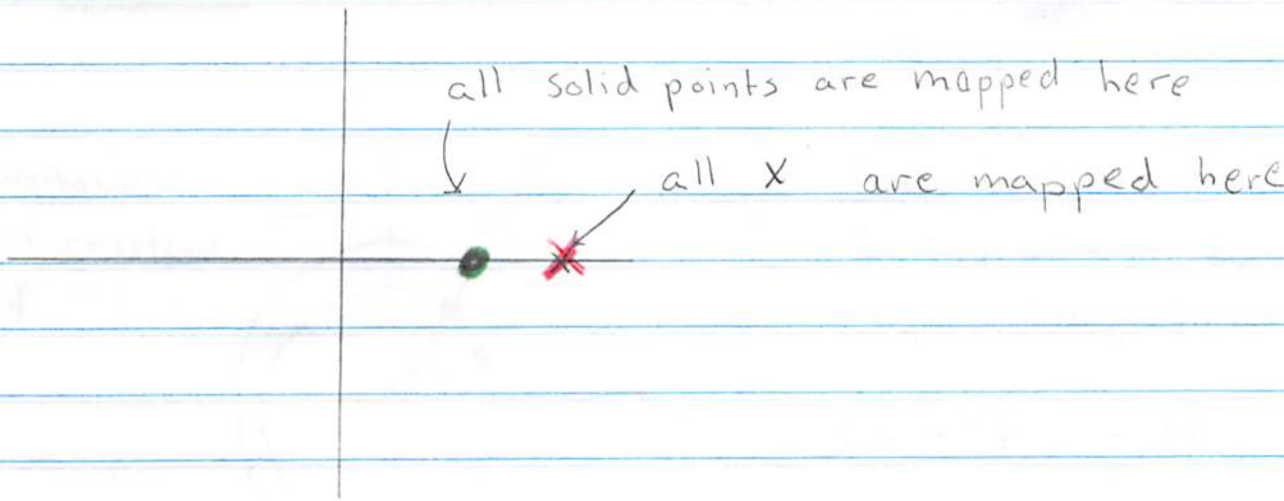
Example 3

$$f(x,y) = x^2 + y^2 = r^2$$



domain — whole complex plain

image — real line



Clearly this example is quite different from the first two.

As we will see it is because the first two examples are holomorphic

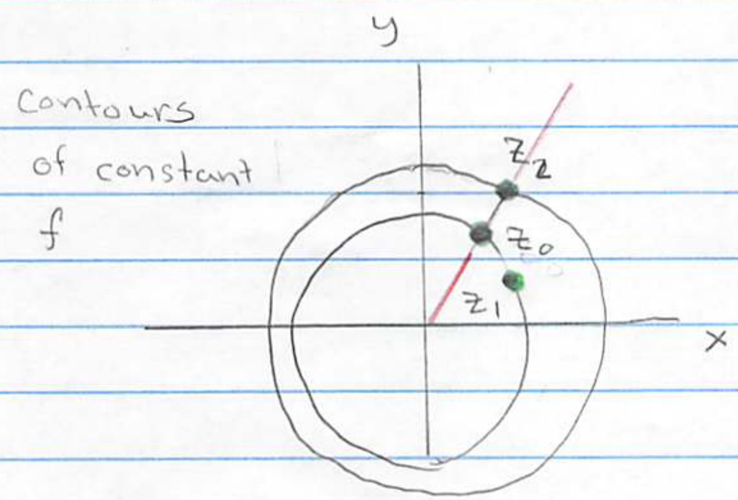
Holomorphic (or Complex Differentiable) functions

- A function is said to be complex differentiable at a point z_0 if

$$\frac{df}{dz} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}$$

and is independent of how you approach z_0

Example $f(x,y) = x^2 + y^2$ is not holomorphic



For instance, if I approach z_0 from z_1 (along circle)

$\Delta f = f(z_1) - f(z_0) = 0$

But, if I approach z_0 from z_2 (along a line of constant angle)
$\Delta f = 2r\Delta r$.

Thus I get two different answers depending on how I approach z_0

Cauchy Riemann Equations

- Let $\left.\frac{df}{dz}\right|_{z_0}$ exist in a neighborhood of z_0 call it $f'(z_0) = a + bi$
- assume u, v are continuous at at least once-differentiable in a neighborhood of z_0

Then

$$\Delta f = f(z) - f(z_0) \quad , \quad \Delta z = \Delta x + i \Delta y$$

and in a neighborhood of z_0

$$\Delta f = f'(z_0) \Delta z$$

$$= (a + ib)(\Delta x + i \Delta y)$$

$$= (a \Delta x - b \Delta y) + i(b \Delta x + a \Delta y)$$

But $f = u + iv$ so:

$$\Delta f = \left(\frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y\right) + i\left(\frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y\right)$$

Comparing we see that for holomorphic functions

$$a = \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad b = \boxed{-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}}$$

- This is expressed in several ways:

$$f'(z) = a + ib$$

all the same $\longrightarrow$

$$f'(z) = \begin{cases} \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y} & \text{or} \quad \dfrac{\partial u}{\partial x} - i\dfrac{\partial v}{\partial x} = \dfrac{\partial f}{\partial x} \\[2ex] \dfrac{\partial v}{\partial y} + i\dfrac{\partial v}{\partial x} & \quad\quad \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y} = -i\dfrac{\partial f}{\partial y} \end{cases}$$

- The cauchy Riemann Equations are often expressed as follows. Define:

$$\left(\frac{\partial}{\partial z}\right) \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \quad \text{and} \quad \left(\frac{\partial}{\partial \bar{z}}\right) \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

Note $z \equiv x + iy \quad \bar{z} = x - iy$

$$\frac{\partial}{\partial z}(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)(x + iy) = 1$$

- And Similary: $\dfrac{\partial \bar{z}}{\partial z} = 0 \quad \dfrac{\partial z}{\partial \bar{z}} = 0 \quad \dfrac{\partial \bar{z}}{\partial \bar{z}} = 1$

Then

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \neq 0$$

$$\frac{\partial v}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right) \neq 0 \quad \ldots \text{ but}$$

But

$$\frac{\partial}{\partial \bar{z}} f = \frac{\partial}{\partial \bar{z}} (u + iv)$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

this is for a general function

$$\boxed{\frac{\partial f}{\partial \bar{z}} = 0}$$

← for a holomorphic function by cauchy riemann equations

this is for a holomorphic fcn.

And similarly,

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{i}{2}\left(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\boxed{\partial f / \partial z = df(z)/dz}$$

for holomorphic functions

Holomorphy and Calculus

Complex derivatives obey normal rules!

- E.g. if $f(x,y)$ and $g(x,y)$ are holomorphic then so is

$$f(x,y)\, g(x,y)$$

Proof:

$$\frac{\partial}{\partial \bar{z}}(f \cdot g) = \cancelto{0}{\frac{\partial f}{\partial \bar{z}}}\, g + f \cancelto{0}{\frac{\partial g}{\partial \bar{z}}} = 0$$

- Product Rule:

$$\frac{d(f \cdot g)}{dz} = \frac{\partial}{\partial z}(f \cdot g) = \frac{\partial f}{\partial z} g + f \frac{\partial g}{\partial z}$$

$$= \frac{df}{dz} g + f \frac{dg}{dz}$$

- Chain Rule:

$$\frac{d}{dz} f(g(z)) = \frac{f(g(z_0 + \Delta z)) - f(g(z_0))}{\Delta z}$$

$$= \frac{f(g(z_0) + g'(z_0)\Delta z) - f(g(z_0))}{\Delta z}$$

$$= \frac{f'(g(z_0))\, g'(z_0)\Delta z}{\Delta z} = f'(g(z_0))\, g'(z_0)$$

Examples

- $f(x,y) = x^2+y^2$ is not holomorphic

$$= (x+iy)(x-iy)$$

$$= z\bar{z}$$

Then

$$\frac{\partial f}{\partial \bar{z}} = z \leftarrow$$ this is not zero, so the function is not holomorphic.

- $f(x,y) = z^2$

Then

$$\underbrace{\left(\frac{\partial f}{\partial \bar{z}}\right)_z = 0}_{\text{so } f \text{ is holomorphic}} \qquad \frac{df}{dz} = \frac{\partial}{\partial z} f(z) = 2z$$

assumes holomorphy

Note:

$$z^2 = (x^2-y^2) + 2ixy$$

- We said for holomorphic fcns:

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = 2x + 2iy = 2z \quad \text{and}$$

$$\frac{df}{dz} = -i\frac{\partial f}{\partial y} = i2y + 2x = 2z$$

all agree as they should!

The real and imaginary parts of f are harmonic

- Want to show that (u is harmonic) ≡ obeys laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Then use $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0$$

- Similarly show $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

- In general we give an alternate proof

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

So

Since f is holomorphic

$$\nabla^2 f = 4\frac{\partial^2 f}{\partial z \partial \bar{z}} = 4\frac{\partial}{\partial z}\cancelto{0}{\frac{\partial f}{\partial \bar{z}}} = 0$$

So f satisfies the laplace equation and so does its real and imaginary parts.

The nature of holomorphic maps

- See Handout

- Given a displacement in the x, y plane we can find the corresponding displacement in the u, v plane

see arrows on handout

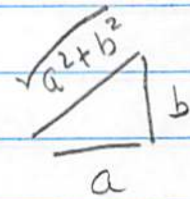
$$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

There are only two independent numbers here. They are given by df/dz

$$\frac{df}{dz} = a + ib = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$$

or

$$\left| \frac{df}{dz} \right| = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

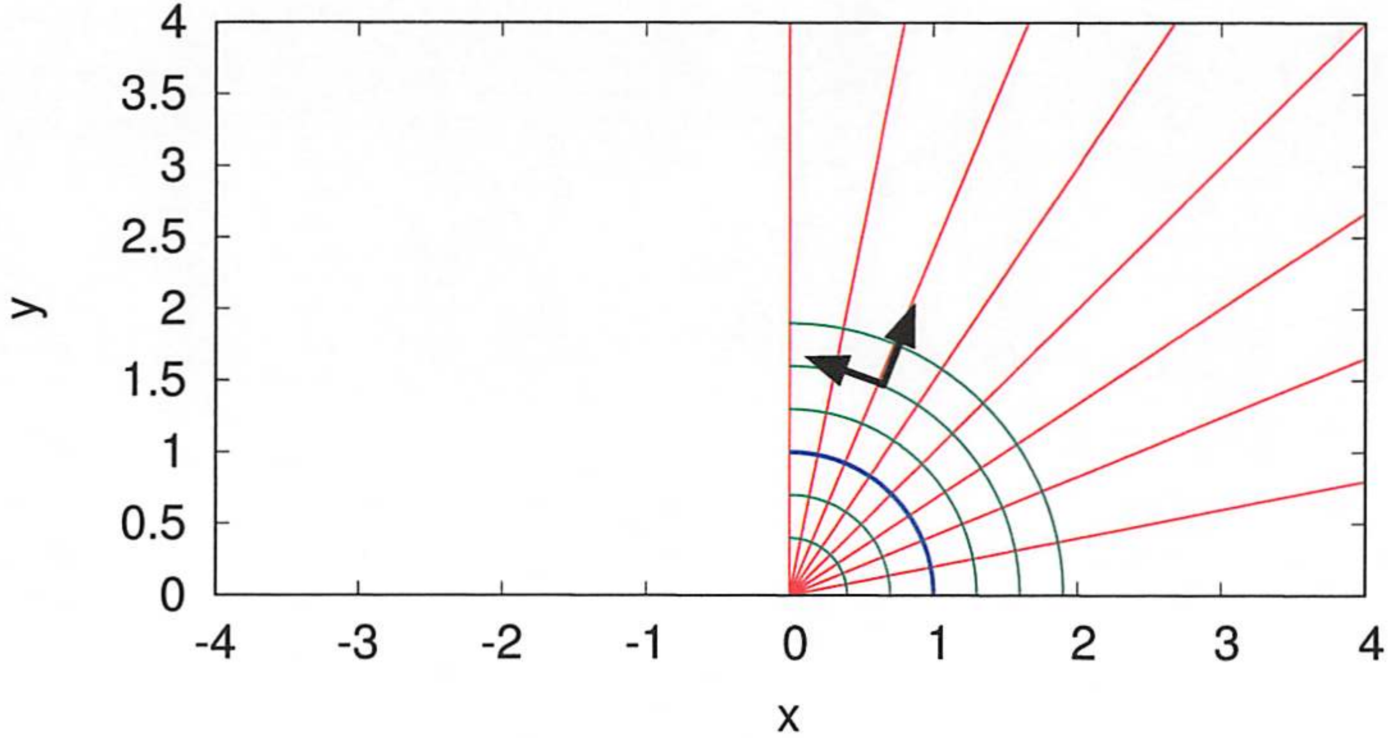


Then

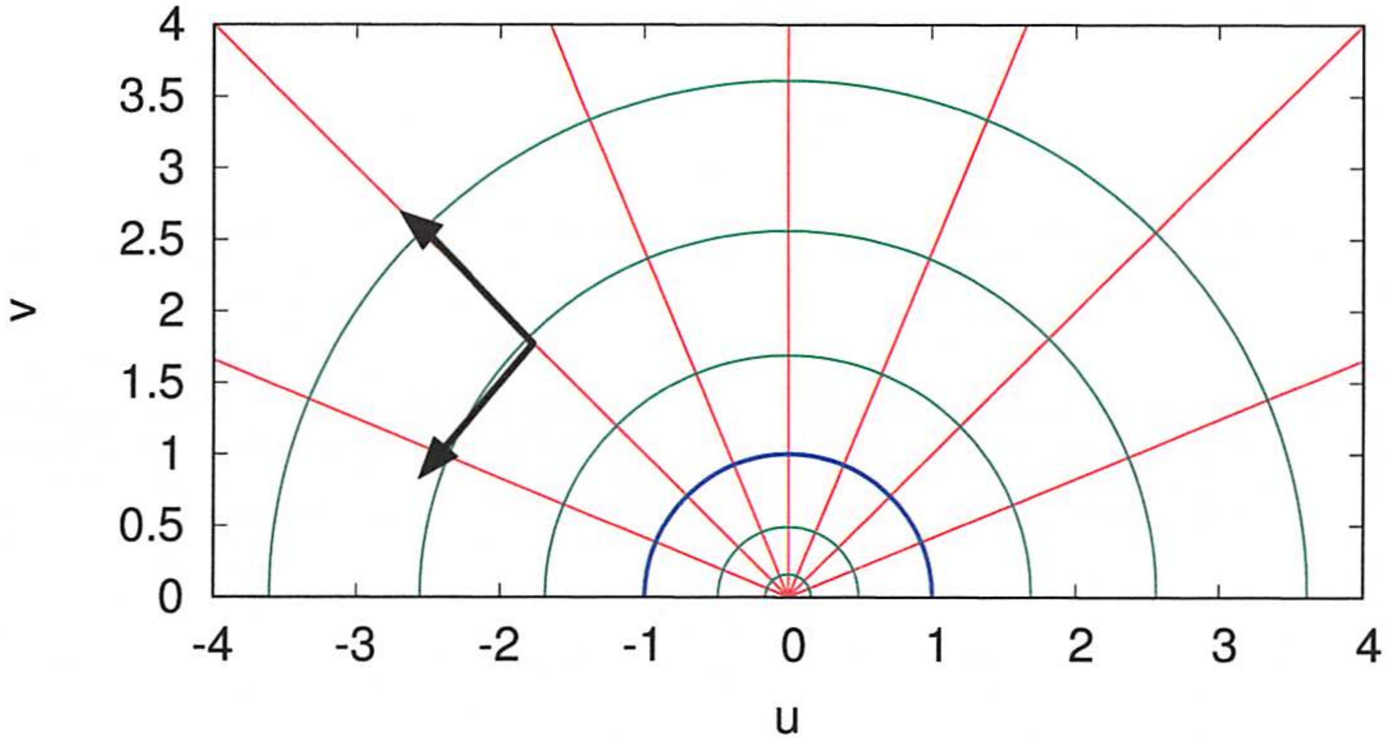
$$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

dividing by $\sqrt{a^2+b^2}$ using $\cos\theta = a/\sqrt{a^2+b^2}$ etc

- $$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = \left| \frac{df}{dz} \right| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$



The map z^2



- Thus we see that the holomorphic map locally just rescales the displacements and rotates the vectors (see picture)

  - Holomorphic Maps preserve the angles between lines, mapping right angles to right angles etc.

Analytic Functions

- A function is analytic in a neighborhood of z_0 if it is described by a (uniformly convergent) power series

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots = \sum_n a_n (z-z_0)^n$$

- Such a fuction is clearly holomorphic since

$$\frac{\partial f}{\partial \bar{z}} = 0$$

- Its derivative is:

$$\frac{df}{dz} = \sum_n n a_n (z-z_0)^{n-1}$$

- We will show the converse, i.e. that holomorphic fcns are analytic

- The most important series is the geometric series

$$f(z) = 1 + z + z^2 + z^3 + \ldots$$

- Then it is easy to see that this series is uniformly convergent for $z<1$:

$$S_n \equiv 1 + z + \ldots + z^n \quad \text{(sum up to } n\text{)}$$

$$S_n = \frac{(1 - z^{n+1})}{(1-z)}$$

Just multiply $(1-z)(1+z+\ldots z^n)$ to prove it.

- For $|z|<1$, $z^{n+1} \longrightarrow 0$ for $n \longrightarrow \infty$ in S_n, and thus:

$$\sum 1 + z + z^2 + \ldots = \frac{1}{1-z}$$