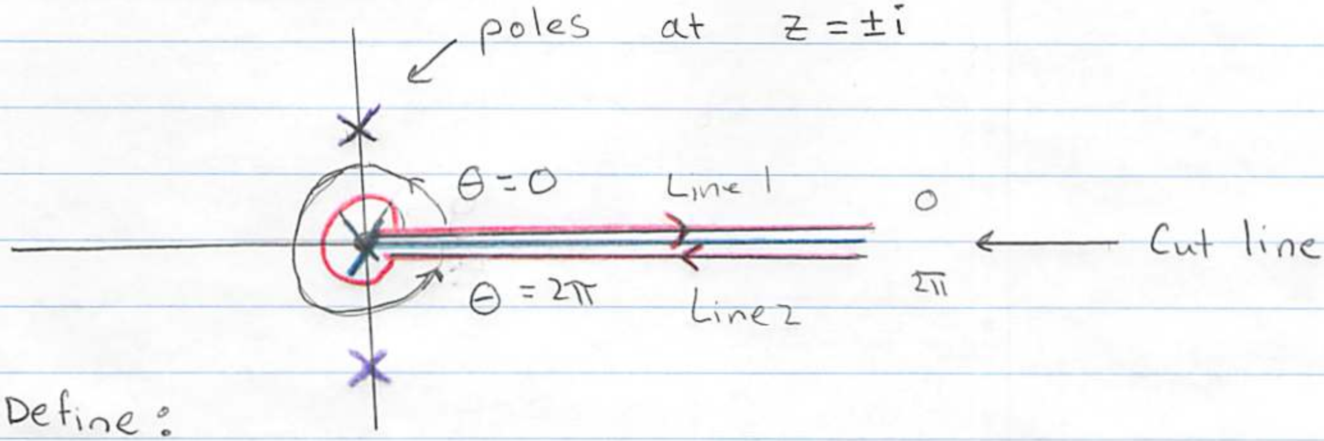
Sample integral



Define:

$$I(\alpha) \equiv \int_0^\infty \frac{1}{x^2+1} x^\alpha \, dx = ? \qquad -1 < \mathrm{Re}\,\alpha < 1$$

$I(0) = \frac{\pi}{2}$

$I(1) =$ divergent

$I(-1) =$ divergent

- Then look at the integrand:

$$f(z) \equiv \frac{z^\alpha}{z^2+1} \qquad f(x) = \frac{x^\alpha}{x^2+1} \quad \text{on Line 1}$$

- $f(z)$ has a branch point at $z = 0$. If I analytically continue the function around $z = 0$ to $\theta = 2\pi$

$$z^\alpha = x^\alpha \quad \text{at } \theta = 0 \longrightarrow f(z) = x^\alpha / x^2+1$$

$$z^\alpha = x^\alpha e^{i 2\pi \alpha} \quad \text{at } \theta = 2\pi \longrightarrow f(z) = e^{2\pi i \alpha} \frac{x^\alpha}{x^2+1}$$

- Then note: because we go backwards on Line 2 (see figure above)

$$I_2 = \int_{\text{Line 2.}} f(z) = -e^{2\pi i \alpha} \int_0^\infty dx \, \frac{x^\alpha}{x^2+1}$$

- So

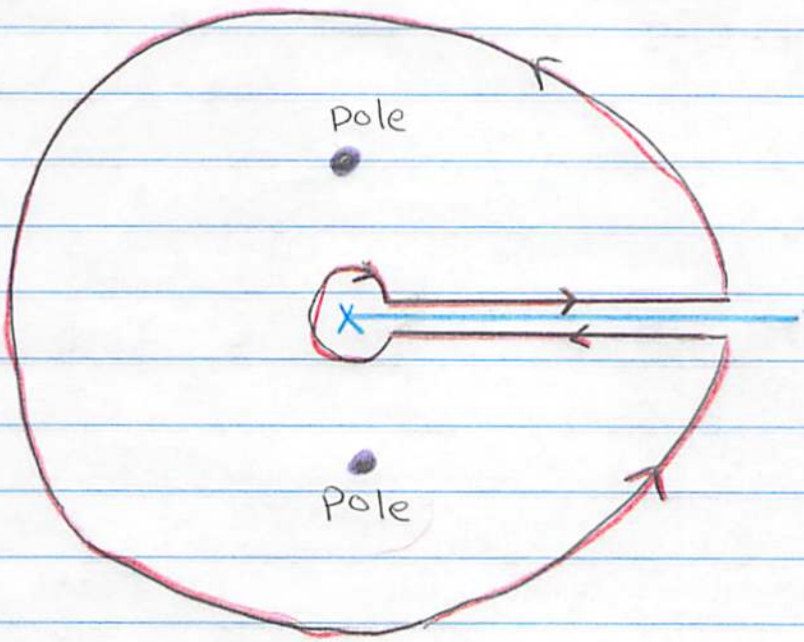
$$\int_{\text{Line 1 + Line 2}} f(z) = (1 - e^{2\pi i \alpha}) \overbrace{\int_0^\infty dx \, \frac{x^\alpha}{(1+x^2)}}^{I(\alpha)}$$

- Now we may close the arcs as follows, using a small circle around the origin, and the great circle

$$\underline{I} \equiv \int_0^\infty dx \, \frac{x^\alpha}{(1+x^2)} = \frac{1}{1-e^{2\pi i \alpha}} \int_{\text{Line 1 + Line 2}} \frac{z^\alpha}{(1+z^2)} \, dz$$

see below for proof

$$= \frac{1}{1-e^{2\pi i \alpha}} \oint_{\text{loop}} \frac{z^\alpha}{(1+z^2)} \, dz$$



Since:

$$f(z) \xrightarrow[z \to \infty]{} \frac{z^\alpha}{z^2} \to 0$$

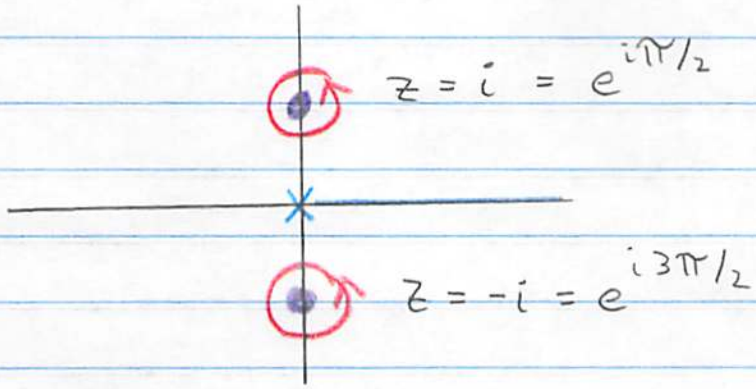
and

$$f(z) \xrightarrow[z \to 0]{} z^\alpha \to 0$$

for $-1 < \operatorname{Re} \alpha < 1$,

- The arcs around the origin and the great circle do not contribute.

- Then we may circle the poles by deforming the contour



- Then the integrals around the poles just gives $2\pi i$ (Res at pole)

$$\operatorname*{Res}_{z=i} \frac{z^\alpha}{(z^2+1)} = \operatorname*{Res} \frac{z^\alpha}{(z+i)(z-i)} = \frac{i^\alpha}{2i} = \frac{e^{i\pi/2\,\alpha}}{2i}$$

And

$$\operatorname*{Res}_{z=-i} \frac{z^\alpha}{(z+i)(z-i)} = \frac{(-i)^\alpha}{-2i} = -\frac{1}{2i} e^{i3\pi/2\,\alpha}$$

- Note the way we chose the branch cut (with $\theta = 0 \ldots 2\pi$) we must take $-i = e^{i3\pi/2}$ rather than $e^{-i\pi/2}$ ← wrong!

Thus

$$I = \frac{1}{1-e^{2\pi i\alpha}} \left[\frac{e^{i\pi\alpha/2}}{2i} - \frac{e^{i3\pi\alpha/2}}{2i} \right]$$

Straightforward algebra gives:

$$I(\alpha) = \frac{\pi \sin \pi\alpha/2}{\sin \alpha\pi}$$

- We can check our results:

For $\alpha = 0$:

$$I(\alpha) = \int_0^\infty \frac{1}{1+x^2}\,dx = \pi/2 \overset{\checkmark}{=} \lim_{\alpha \to 0} \frac{\pi \sin \pi\alpha/2}{\sin \alpha\pi}$$

- Note that as $\alpha \to \pm 1 + \varepsilon$ then

$$I(\pm 1 + \varepsilon) = \pm \frac{1}{\varepsilon}$$

Showing as expected that

$$I(\alpha) = \int_0^\infty \frac{x^\alpha}{(1+x^2)}\,dx \quad \text{diverges as} \quad \alpha \to \pm 1$$

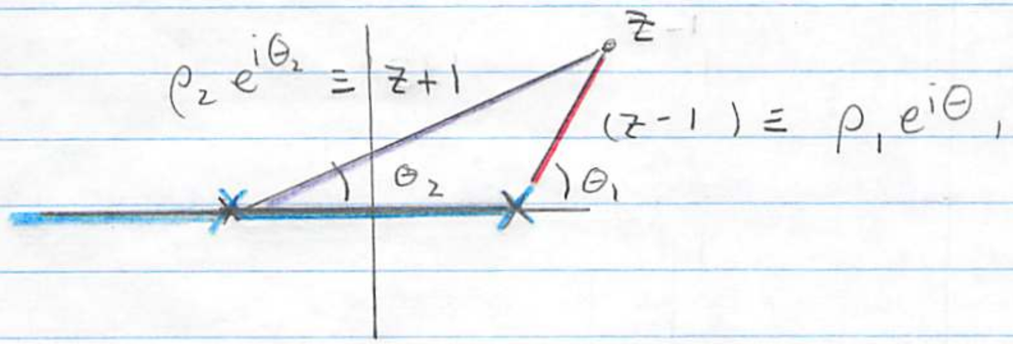
- The expression

$$I(\alpha) = \frac{\pi \sin(\pi\alpha/2)}{\sin(\alpha\pi)}$$

provides an analytic continuation of $I(\alpha)$ throughout the complex plane

Another Example - Of analytic continuation

Take $\sqrt{1-z^2}$ this is $\sqrt{1-z}\,\sqrt{1+z}$.
It has branch pooints at $z=1$ and -1.
We place a cut from $(-\infty,-1)$ and $(-\infty,1)$

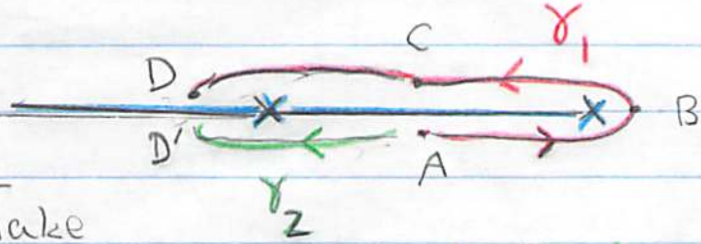


Then we will define this function throughout the plane:

$$\sqrt{(1-z)^2} = \sqrt{(-1)}\ \sqrt{z-1}\ \sqrt{z+1}$$

$$= e^{i\pi/2}\sqrt{\rho_1\rho_2}\ e^{i(\theta_1+\theta_2)/2}$$

Take a starting point A, and conventionally define its value. Then our branch choices conventionally define $\sqrt{1-z^2}$, by analytic continuation along paths not crossing the cut:



Take path γ_1	θ_1	θ_2	$\sqrt{1-z^2}$
A	$-\pi$	≈ 0	$\sqrt{\rho_1\rho_2}$
B	0	$\simeq 0$	$\sqrt{\rho_1\rho_2}\ e^{i\pi/2} = \sqrt{\rho_1\rho_2}\ i$
C	$+\pi$	0	$\sqrt{\rho_1\rho_2}\ e^{i\pi/2}e^{i\pi/2} = -\sqrt{\rho_1\rho_2}$

starting point essentially by definition

Take path γ_1

	θ_1	θ_2	$\sqrt{1-z^2}$
D	$+\pi$	$+\pi$	$\sqrt{\rho_1\rho_2}\; e^{i\pi/2} e^{i\pi/2} e^{i\pi/2} = \sqrt{\rho_1\rho_2}\,(-i)$
D′	$-\pi$	$-\pi$	$\sqrt{\rho_1\rho_2}\; e^{i\pi/2} e^{-i\pi/2} e^{-i\pi/2} = \sqrt{\rho_1\rho_2}\,(-i)$

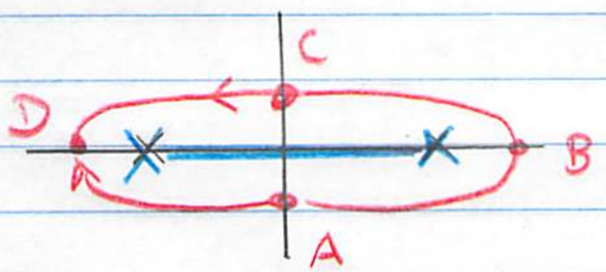
take path γ_2
(see figure above)

Then notice:

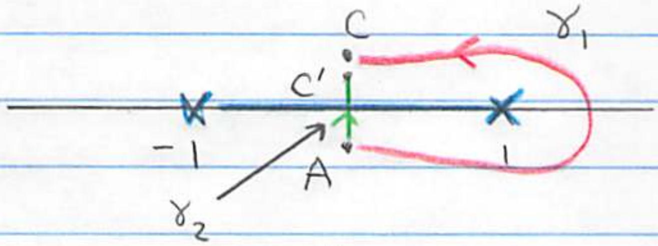
- A, C differ in sign because the path γ_1 encircles the branch point once.

☆☆ • $D = D'$ because the combined path γ_1, γ_2 encircles two branch points of the sqrt type. Each encircling yields $e^{i2\pi/2}$ for a total phase of $e^{i2\pi/2} \cdot e^{i2\pi/2} = 1$

- Thus we may define a single valued function on the set with the deleted segment $[-1 \ldots 1]$



There is no rule that says that we can't analytically continue across the cut from our starting point (point A)



or previously agreed upon

The cut defines a canonical value of $\sqrt{1-z^2}$ starting from A. If we analytically continue directly from A to C' along γ_2 we find

		θ_1	θ_2	$\sqrt{1-z^2}$
via γ_1	C	$+\pi$	$\simeq 0$	$-\sqrt{\rho_1\rho_2}$
via γ_2	C'	$-\pi-\varepsilon$	$\simeq 0$	$\sqrt{\rho_1\rho_2}\, e^{i\pi/2}\, e^{-i\pi/2-i\varepsilon} \simeq \sqrt{\rho_1\rho_2}$.

this differs by a sign from A

small

this is the same as A
we have only analytically continued a little bit, order ε, along γ_2