Boundary Conditions and The Inhomogeneous Equations

- The generic general solution to the inhomogeneous equation can be written

$$y(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x)}_{\equiv y_{homo}(x)} + y_p(x) \qquad (12.9)$$

homogeneous sol

the particular solution

Where the particular solution is one solution to $\mathcal{L} y_p(x) = f(x)$ and $y_1(x)$ and $y_2(x)$ solve the homogeneous equation $\mathcal{L} y(x) = 0$

- To prove this, suppose I have two inhomogeneous solutions to $\mathcal{L} y_p(x) = f(x)$, $y_{p1}(x)$ and $y_{p2}(x)$.

Their difference $\Delta y = y_{p1} - y_{p2}$ satisfies

$\mathcal{L} \Delta y = 0$ and thus can be written as

$$\Delta y = c_1 y_1 + c_2 y_2 \leftarrow \text{a general solution to } \mathcal{L} y = 0$$

- Further, if the boundary conditions are inhomogeneous (e.g. $v(t)|_{t=0} = v_0$) we can always let the homogeneous solution solve the inhomogeneous b.c. Then the particular solution satisfies the homogeneous b.c..

For example, if the b.c. are for the full system are:

$$\left.\begin{aligned} \alpha_{11}\, y(a) + \alpha_{12}\, y'(a) &= A \\ \alpha_{21}\, y(a) + \alpha_{22}\, y'(a) &= B \end{aligned}\right\} \quad \text{In homogeneous b.c} \qquad (13.1)$$

By making $y_{homo}(x)$ satisfy Eq (13.1) (by adjusting c_1 and c_2), then in order for the combination $y(x) = y_{homo}(x) + y_p(x)$ to satisfy Eq (13.1) we:

must demand that the particular solution satisfy

$$\left.\begin{aligned} \alpha_{11}\, y_p(a) + \alpha_{12}\, y_p'(a) &= 0 \\ \alpha_{21}\, y_p(a) + \alpha_{22}\, y_p'(a) &= 0 \end{aligned}\right\} \quad \text{The homogeneous version of Eq (13.1)}$$

This is almost always the best way to procede.

Green Functions

- We may formally solve the problem of finding a particular solution, by finding a function $G(x, x_0)$ which satisfies

$$\mathcal{L}\, G(x, x_0) = \delta(x - x_0)$$

i.e.

$$\left[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)\right] G(x, x_0) = \delta(x - x_0)$$

or

$$\left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] G(x, x_0) = \delta(x - x_0)$$

- Typically we work in an interval $[a,..b]$ and demand (as disussed) in the previous section) that G satisfies homogeneous b.c. at a, b

Then the particular solution is

$$y_p(x) = \int_a^b dx_0\, G(x, x_0)\, f(x_0)$$

- Then we see that this satisfies the inhomo eq $\mathcal{L} y_p = f$

$$\mathcal{L}_x y_p(x) = \int_a^b dx_0\, \mathcal{L}_x G(x, x_0)\, f(x_0)$$

$$= \int_a^b dx_0\, \delta(x - x_0)\, f(x_0) = f(x)$$

Constructing the Green Fcn

- Away from x_0, $G(x, x_0)$ satisfies the homogeneous equation

$$\mathcal{L}\, G(x, x_0) = 0 \quad \text{for } x \neq x_0$$

- The Green function is found by solving the homogeneous equation for $x > x_0$ and $x < x_0$ and matching the solutions by integrating the equation across the δ-fcn.

Example: Consider a particle subject to drag and a force which acts for $t > 0$

(15.1) $$\frac{dv}{dt} + \eta v = f(t) \qquad v(0) = v_0 \qquad t \in [0, \infty]$$

an inhomo b.c.

- We first set $f(t) = 0$ and solve for the homogeneous solution satisfying the inhomo b.c

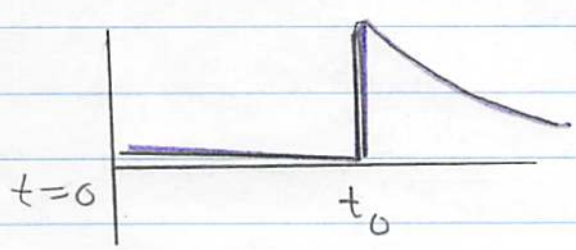
(15.2) $$v(t) = v_0 e^{-\eta t}$$

- Then we solve the inhomo equation with

(15.3) $$\frac{dG}{dt} + \eta G = \delta(t - t_0) \qquad G\Big|_{t=0} = 0$$

← homogeneous b.c.

☆ First, qualitatively, this is a partice at rest, which gets an implusive kick at t_0:



we will show

$$G(t, t_0) = \Theta(t - t_0)\, e^{-\eta(t - t_0)}$$

The particle gets a kick at $t = t_0$, its velocity then decays fro $t > t_0$.

For $G(t, t_0)$ we

(1) Solve (15.1) for $t > t_0$, (2) Solve for $t < t_0$, and (3) match.

(1) The general homogeneous sol is $C_1 e^{-\eta t}$ for $t > t_0$

(2) The general homogeneous sol for $t < t_0$ is

$C_2 e^{-\eta t}$. But we must set $C_2 = 0$ to

satisfy the homogeneous b.c. $G(t)\Big|_{t=0} = 0$

(3) We then match the two solutions.
Integrating the EOM from $t_0 - \varepsilon$ to $t_0 + \varepsilon$, and demanding that G remains bounded we find

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} dt \left[\frac{dG}{dt} + \eta G \right] = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0)$$

$$G\Big|_{t_0+\varepsilon} - G\Big|_{t_0-\varepsilon} + O(\varepsilon) = 1$$

this is small $\int_{t_0-\varepsilon}^{t_0+\varepsilon} G$ for bounded G

In the current case

$G(t, t_0)\Big|_{t=t_0-\varepsilon} = 0$ and therefor

$$G(t, t_0)\Big|_{t=t_0+\varepsilon} = 1$$

From our known solution, $G(t_0+\varepsilon) = Ce^{-\eta t}\Big|_{t=t_0}$,
for $t > t_0$ we find

$$Ce^{-\eta t_0} = 1 \Rightarrow C = e^{+\eta t_0} \qquad \text{and}$$

thus

$$G(t) = e^{-\eta(t-t_0)} \qquad t > t_0$$

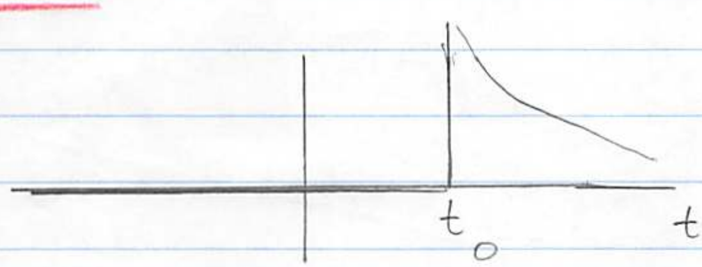
Or

$$\boxed{G(t,t_0) = \Theta(t-t_0)\, e^{-\eta(t-t_0)}}$$

unit impulsive force

$$\frac{dv}{dt} + \eta v = \delta(t-t_0)$$

Picture:

(16.1) 

velocity jumps, and then decays

It is easy to check that $\mathcal{L}\, G(t,t_0) = \delta(t-t_0)$.
The δ-fcn arises when we differentiate

$$\boxed{\frac{d}{dt}\Theta(t-t_0) = \delta(t-t_0)} \leftarrow \text{origin of } \delta\text{-fcn in } \mathcal{L}G = \delta(t-t_0)$$

The General Solution:

$$y(t) = y_{homo}(t) + \int_a^\infty G(t,t_0)\, f(t_0)\, dt_0$$

$$y(t) = Ce^{-\eta t} + \int_a^t e^{-\eta(t-t_0)} f(t_0)\, dt_0$$

Second Order DEQ and Green fcns

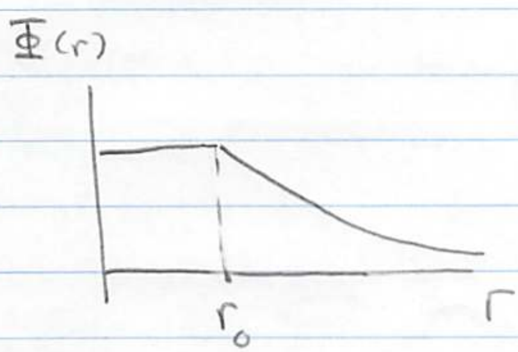
- The procedure to constructi the green function is always the same as in the previous example
  - Solve the homogeneous equations to the left and right
  - integrate across the δ-fcn to match the two sols

Let us follow this procedure for the equation

$$\left[-\frac{d}{dx}\, p(x)\, \frac{d}{dx} + q(x) \right] G(x,x_0) = \delta(x - x_0)$$

- For definiteness take a specific problem. The potential of a charged spherical shell

$$\Phi(r, r_0) = \begin{cases} \dfrac{Q}{4\pi r_0} & r < r_0 \\[2ex] \dfrac{Q}{4\pi r} & r > r_0 \end{cases}$$



This is the solution to

$$\nabla \cdot E = \rho \qquad E = -\nabla \Phi$$

$$-\nabla^2 \Phi = \rho$$

spherical shell of radius r_0 and charge Q

Where $\rho(r) = \dfrac{Q}{4\pi r^2}\,\delta(r-r_0)$

$$-\nabla^2 \Phi = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \Phi}{\partial r} = \frac{Q}{4\pi}\frac{1}{r^2}\,\delta(r-r_0)$$

So we see that we are trying to find the Green function of

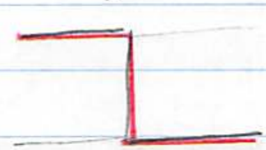
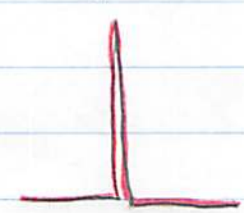
$$-\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} G(r,r_0) = \delta(r-r_0) \qquad \Phi = \frac{Q}{4\pi} G(r,r_0)$$

- Note we are looking for a solution that looks like this:

δ-fcn

θ fcn

kink

$y(x) =$  $y'(x) =$  $y''(x) =$ 

So the function is continuous, but has a discontinuous derivative. We can see this directly from the general EOM

Integrating from $x_0 - \varepsilon$ to $x_0 + \varepsilon$

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} dx \left[-\frac{d}{dx}\left(p(x) \frac{dG}{dx} \right) + q(x)\, G \right] = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x - x_0)$$

Yielding

electric field in our example

$$- p(x) \frac{dG}{dx} \Bigg|_{x = x_0 + \varepsilon} + p(x) \frac{dG}{dx} \Bigg|_{x = x_0 - \varepsilon} = 1 \qquad (20.1)$$

Jump condition

For the simple case of a charged sphere this says that the jump in the electric field is due to the surface charge.

$$-r^2 \frac{\partial G}{\partial r} \Bigg|_{r = r_0 + \varepsilon} - \left(-r^2 \frac{\partial G}{\partial r} \right) \Bigg|_{r = r_0 - \varepsilon} = 1 \qquad (20.2)$$

or since $G = \Phi / (Q/4\pi)$ and $E_r = -\partial \Phi / \partial r$

$$4\pi r^2 E_r \Big|_{out} - 4\pi r^2 E_r \Big|_{in} = Q \qquad (20.3)$$

- Now we have analyzed the "jump condition" which relates the interior and exterior solutions. We now should solve for these solutions

- Let the homogeneous solution to the left $(x < x_0)$ be $y_<(x)$ and the solution to the right be $y_>(x)$ (i.e $x > x_0$). For our case we solve

$$-\frac{d}{\partial r} r^2 \frac{\partial}{\partial r} \Phi = 0 \qquad \left\{ \begin{array}{l} \text{b.c. regular as } r \to 0 \\ \text{and vanishes as } r \to \infty \end{array} \right.$$

There are two solutions 1 and $1/r$

$$y(r) = C_1 \cdot 1 + C_2 \cdot \frac{1}{r}$$

- Our solutions obey homogeneous b.c. (why?) So one of these constants can be chosen at will, since if y satisfies the DEQ and b.c. then so does $c\,y(r)$. Here then, $y_<(r) = 1$ and $y_>(r) = \frac{1}{r}$

These follow from our b.c., that $G(r, r_0)$ be regular ar $r \to 0$ and $r \to \infty$.

- The Green function then takes the form

$$G(x, x_0) = C_1\, y_<(x)\, \Theta(x_0 - x) + C_2\, y_>(x)\, \Theta(x - x_0)$$

Continuity Givers at $r = r_0$ the condition:

(21.1) $$G(x, x_0) = C\left[y_<(x)\, y_>(x_0)\, \Theta(x_0 - x) + y_>(x)\, y_<(x_0)\, \Theta(x - x_0) \right]$$

There is a an interesting point here which we will return to later. When specifying the boundary conditions for the solutions of

$$\frac{-\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)$$

The function $p(r) = r^2$ has zeros and singularities at $r = 0$ and $r = \infty$. These are known as singular points of the diffeq. The two solutions are

$$y_1(r) = 1 \quad y_2(r) = \frac{1}{r}$$

which are regular as $r \to 0$ and $r \to \infty$ respectively. Our boundary conditions are regularity at $r = 0$ and $r = \infty$. These boundary conditions are homogeneous. For instance if $y_1 = C_1/r$ and $y_2 = C_2/r$ are regulars as $r \to \infty$ then any linear combination is also regular as $r \to \infty$

- Now we can determine the remaining coefficient C from the jump condition Eq (20.1)

Substituting Eq (21.1) into Eq (20.1) gives
(Do it!!!)

$$-p(x)\, C\, y_>'(x)\, y_<(x_0)\Big|_{x=x_0+\varepsilon} + p(x)\, C\, y_<'(x)\, y_>(x_0)\Big|_{x=x_0-\varepsilon} = 1$$

Or

$$C = \frac{1}{p(x_0)\left[y_>(x_0)\, y_<'(x_0) - y_>'(x_0)\, y_<(x_0) \right]}$$

i.e.

$$C = \frac{1}{p(x_0)\, W(x_0)} \qquad \underbrace{W(x_0) = y_>\, y_<' - y_<\, y_>'}_{\text{Wronskian of } y_>, y_<}$$

recall that this is constant (Eq 6.1)

and

$$G(x, x_0) = \frac{y_>(x)\, y_<(x_0)\, \Theta(x - x_0) + y_<(x)\, y_>(x_0)\, \Theta(x_0 - x)}{p(x_0)\, W(x_0)}$$

Eq (22.1)

the green fcn for a general 2nd order DEQ.

the denom is a constant indep of x_0

- For our particular example

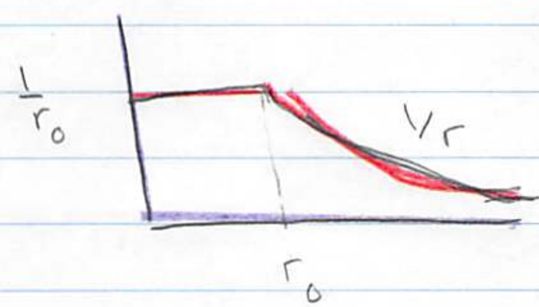
$$y_> = 1/r \qquad y_< = 1 \qquad p(r) = r^2$$

$$W(r) = \cancelto{0}{y_> y_<'} - y_< y_>' = -1 \cdot \frac{\partial}{\partial r}\frac{1}{r} = \frac{1}{r^2}$$

And $p(r)\,W(r) = 1$ thus

$$G(r, r_0) = \frac{1}{r} \cdot 1 \; \Theta(r - r_0) + 1 \cdot \frac{1}{r_0} \; \Theta(r_0 - r) \quad \checkmark$$

- Graph



as expected! This is the potential of a charged sphere up to $Q/4\pi$

We said that for the differential equation

$$\left[-\frac{d}{dx}p(x)\frac{d}{dx}+q(x)\right]G(x,x_o)=\delta(x-x_0)$$

we had a green function given by Eq. (22.1) in the hand written notes. Expanding the above equation, we have

$$-p(x)\left[\frac{d^2}{dx^2}+P(x)\frac{d}{dx}+Q(x)\right]G(x,x_0)=\delta(x-x_0)$$

were $P(x) = p'(x)/p(x)$ and $Q(x) = q(x)/(-p(x))$. Now note that the green function $\hat{G}$ of Eq. (1) in the notes satisfies

$$\left[\frac{d^2}{dx^2}+P(x)\frac{d}{dx}+Q(x)\right]G_{\rm other}(x,x_o)=\delta(x-x_0)$$

and is therefore just a little different from Eq. (22.1) because the coefficient of d^2/dx^2 is normalized differently. Thinking carefully, if we know G then multiplying the "other" equation by $-p(x)$ and changing $-p(x)\to-p(x_0)$ on the rhs where it multiplies $\delta(x-x_0)$, we can conclude that

$$-p(x_0)G(x,x_0)=G_{\rm other}(x,x_0)$$

For this reason the green function sometimes looks different in different books.

Green Fcn of the SHO

$$\left[m\frac{d^2}{dt^2} + \eta m \frac{d}{dt} + m\omega_0^2 \right] x = f(t) \qquad \begin{array}{l} x(a) = x_0 \\ v(a) = v_0 \end{array}$$

- First the homogeneous equations. The general procedure for differential equations is to look for solutions of the form, $e^{-i\omega t}$, and to solve for ω

$$\left[-m\omega^2 - im\omega\eta + m\omega_0^2 \right] e^{-i\omega t} = 0 \qquad \begin{array}{l} x(a) = x_0 \\ v(a) = v_0 \end{array}$$

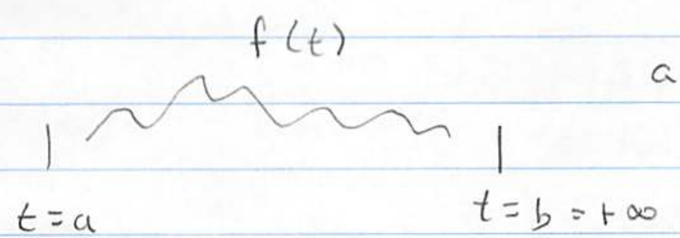
Solve for small η, $\omega = \pm\omega_0 - i\frac{\eta}{2}$

So the homogeneous solution is

we adjust A_1 and A_2 to satisfy b.c.

$$x(t) = A_1 e^{-\eta/2\, t} e^{i\omega_0 t} + A_2 e^{-\eta/2\, t} e^{-i\omega_0 t}$$

- Now we look for the Green function:



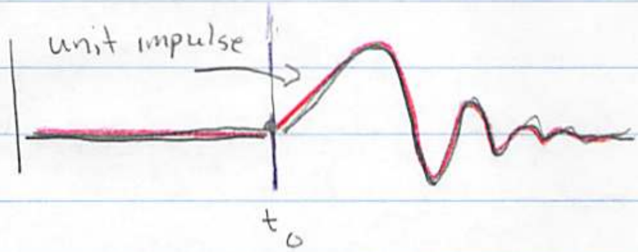
$a < t_0$

causal b.c. homogeneous

$$G(a, t_0) = 0$$
$$\dot{G}(a, t_0) = 0$$

$$\mathcal{L}\, G(t, t_0) = \delta(t - t_0)$$

Then qualitatively: the particle gets a kick at t_0 and then moves



① Then $x_<(t) = 0$. This is the only homogeneous solution which satisfies the b.c. $G(a, t_0) = 0$ $\frac{dG}{dt}(a, t_0) = 0$

② For $x_>(t)$ we take a super-position of the two exponents which vanishes at t_0

$$x_>(t) = C e^{-\gamma/2 (t-t_0)} \sin(\omega_0 (t - t_0))$$

③ The remaining constant follows from the jump

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} m\frac{d^2 x}{dt^2} + \underbrace{\text{lower derivs}}_{0} = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t-t_0)$$

$$m\frac{dx_>}{dt}\Big|_{t_0+\varepsilon} - \cancel{m\frac{dx_<}{dt}}\Big|_{t_0-\varepsilon} = 1$$

unit impulse $\rightarrow$ $m\frac{dx_>}{dt}\Big|_{t_0+\varepsilon} = 1$ $\Big\}$ $C = \frac{1}{m\omega_0}$

$$x_>(t) = C\omega_0 (t-t_0) + O((t-t_0)^2)$$

So

$$(26.1) \quad G(t,t_0) = \frac{e^{-\eta/2(t-t_0)}}{m\omega_0} \sin(\omega_0(t-t_0)) \quad \Theta(t-t_0)$$

The general solution is then

$$y(t) = A_1 e^{-\eta/2 t} e^{+i\omega_0 t} + A_2 e^{-\eta/2 t} e^{-i\omega_0 t} + \int_a^t f(t_0) \frac{e^{-\eta/2(t-t_0)}}{m\omega_0} \sin(\omega_0(t-t_0))\, dt_0$$

Causal Green Functions via Fourier Transform

- For very simple equations like the damped SHO or the drag equation the F.Transfrorm is an easy way to procede

$$\frac{dv}{dt} + \eta v = f(t) \qquad v(a) = V_0$$

- with homogeneous solution $Ce^{-\eta t}$ and Green function satisfying

$$\frac{dG}{dt} + \eta G = \delta(t-t_0) \qquad G(t=a, t_0) = 0$$