Take the Legendre DEQ as Example for $z \to \infty$

$$\left[ +\frac{\partial}{\partial x} \underset{\substack{\uparrow \\ p(x)}}{(1-x^2)} \frac{\partial}{\partial x} + \nu(\nu+1) \right] y = 0$$

- Has regular singular points at $x = \pm 1$ where $p(x)$ vanishes. Now lets determine the behaviour at $\infty$

$$\left[ +\frac{\partial}{\partial x} \frac{(1-x^2)}{x} x \frac{\partial}{\partial x} + \nu(\nu+1) \right] y = 0$$

- let $w = 1/x$

$$\left[ -w^2 \frac{\partial}{\partial w} \left( \left(1 - \frac{1}{w^2}\right) w^2 \frac{\partial}{\partial w} \right) + \nu(\nu+1) \right] y = 0$$

or taking $w \to 0$:

$$\left[ -\frac{\partial^2}{\partial w^2} + \frac{\nu(\nu+1)}{w^2} \right] y = 0$$

- Thus we see that Legendre has regular singular points at $z = \pm 1, \infty$. The class of functions with three regular singular points is known as Hypergeometric functions. To learn about them study the Legendre equation.

Lets Determine the indices (singular behaviour) of the Legendre Equation

$$\left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + \nu(\nu+1)\right]y = 0$$

Near $x=1$ we multiply by $(1-x)$ and approximate $(1-x^2) \simeq 2(1-x)$, Find

$$\left[(1-x)\frac{d}{dx}2(1-x)\frac{d}{dx} + \nu(\nu+1)(1-x)\right]y = 0$$

small as $x \to 1$

So substituting $(1-x)^s$ find

$$(1-x)\frac{d}{dx}(1-x)^s = -s(1-x)^s$$

And

$$s^2(1-x)^s = 0 \qquad s_1 = s_2 = 0$$

In accord with our discussion of Euler equations

or Equidimensial

$$y = C_1 + C_2 \log(1-x)$$

In general a series solution near $x = \pm 1$ gives

$$y = C_1 \overbrace{P_\nu(x)}^{\text{legendre fcn of 1st kind}} + \underbrace{C_2 Q_\nu(x)}$$

approaches $\log(1-x)$

appräches constant as $x \to 1$

Legendre function of 2nd kind

Now lets determine the behavior as $z \to \infty$

From

$$\left[ -\frac{\partial^2}{\partial w^2} + \frac{\nu(\nu+1)}{w^2} \right] y = 0$$

- we substitute $w^s$

$$+ s(s-1) = \nu(\nu+1) \qquad s_1 = -\nu \qquad s_2 = 1+\nu$$

- Take $\nu = 1$ for definiteness ; we found

$$y(x) = C_1 w^{-1} + C_2 w^2 = C_1 z + C_2 \frac{1}{z^2}$$

irregular at $z \to \infty$ — regular as $z \to \infty$

In general

$$y(z) = C_1 P_1(z) + C_2 Q_\nu(z)$$

irregular at $\infty$ — regular as $z \to \infty$

- i.e. the regular and irregular solutions have changed roles as $z$ goes from 1 to $\infty$

This is the expected behaviour

## Essential Singularities and Irregular Sing Points

- Take

$$\left[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)\right] y(x) = 0$$

If $P(x)$ diverges faster than $1/(x-x_0)$ or $Q(x)$ diverges faster than $1/(x-x_0)^2$. Then $x_0$ is called an irreg singular point, and is often associated with an essential singularity

Example

$$\left[-\frac{d^2}{dz^2} + k^2\right] y = 0 \qquad y_{\pm} = e^{\pm kz} \qquad (42.1)$$

- Now as $z \to \infty$ or $w \equiv 1/z$ near zero $e^{\pm k/w}$ has an essential singularity, i.e. its laurent series as an infinite number of neg terms

$$e^{+k/w} \simeq 1 + \frac{k}{w} + \frac{k}{w^2}\frac{1}{2!} + \ldots$$

- From the DEQ, eq (42.1), we find $\bar{y}(w) = y(z)$ satisfies:

$$\frac{d}{dz} = -w^2\frac{d}{dw^2}$$

$$\left[-\frac{d^2}{dw^2} - \frac{2}{w}\frac{d}{dw} + \frac{k^2}{w^4}\right]\bar{y} = 0$$

irreg sing point

Take Bessel's Eqn Again and essential singularity at $\infty$

$$\left[ x\frac{d}{dx}\, x\frac{d}{dx} - (\nu^2 - x^2) \right] y = 0$$

- Becomes for $x = z = 1/w$

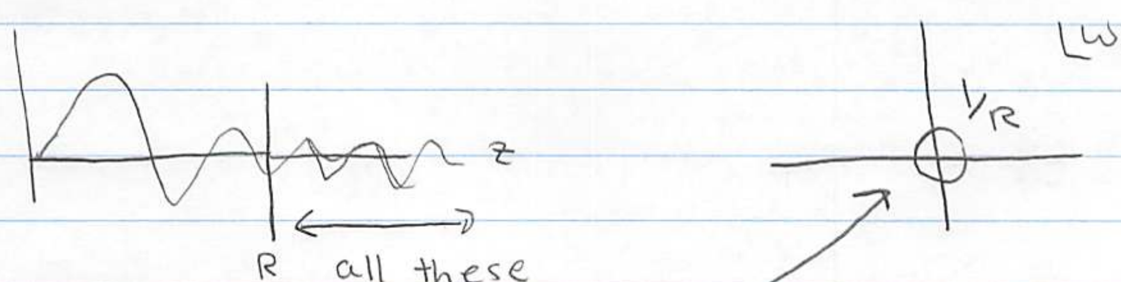
$$\left[ w\frac{d}{dw}\, w\frac{d}{dw} - \left(\nu^2 - \frac{1}{w^2}\right) \right] \bar{y} = 0$$

- Dividing by $w^2$

$$\left[ \frac{1}{w}\frac{d}{dw}\, w\frac{d}{dw} - \left(\frac{\nu^2}{w^2} - \frac{1}{w^4}\right) \right] \bar{y} = 0$$

- Thus we see that the Bessel Eqn has an essential singularity as $z \to \infty$

- The oscillations as $z \to \infty$




oscillations are mapped by $w = 1/z$ into a circle of radius $1/R$. This point $w = 0$ is thus rather irregular

- We now see that the Bessel Eqn has a reg-sing point at $x = 0$ (see previous lectures) and an essential singularity at $x \to \infty$

It is an example of a class of functions with this (i.e reg-sing $x = 0$ + irreg-sing $x = \infty$) property, known as confluent hypergeometric functions

To learn about these functions study Bessel's equation and functions

## Perturbative solutions when $s_1 - s_2 = 0, 1, 2, 3 \ldots$

Previously we constructed for $s_1 - s_2 \neq 0, 1, 2, 3$ (with $s_1 \geq s_2$ by definition) perturbative solutions

$$y_1 = x^{s_1}(1 + a_1 x + a_2 x^2 + \ldots)$$

$$y_2 = x^{s_2}(1 + b_1 x + b_2 x^2 + \ldots)$$

If $s_1 = s_2$ then these solutions are not distinct
If $s_1 - s_2$ positive integer then $y_1$ and $y_2$ both look like $x^{s_1} x^n$ after a finite number of terms

When $s_1 = s_2$, and usually when $s_1 - s_2 =$ pos int

$$y_2(x) = \ln x \, y_1(x) + C x^{s_2}(1 + b_1 x + b_2 x^2 + \ldots)$$

Though in a few cases (e.g. $J_{3/2}(x)$ and $J_{-3/2}(x)$) the solution may work, e.g. for $J_{-3/2}$ with $s_2 = -3/2$

$$y_2(x) = C x^{s_2}(1 + b_1 x + b_2 x^2 + \ldots)$$ ← works fine for $J_{-3/2}$

↳ This is known as Fuch's theorem

Let us see how this works for Bessel's eqn for $\nu = 0$ near $x = 0$

$$\left[ x \frac{d}{dx} x \frac{d}{dx} + x^2 \right] y = 0$$

small near $x = 0$

We set up a perturbative expansion again

$$y = y^{(0)} + y^{(2)} + y^{(4)} + \dots$$

$\longleftarrow$ the odd terms vanish

<u>0th</u>

$$\left[x \frac{d}{dx} x \frac{d}{dx}\right] y^{(0)} = 0 \qquad \text{Subs } x^s, \quad s^2 = 0$$

Then $\quad y^{(0)} = C_1 + C_2 \ln x$

<u>2nd</u>

$$\left[x \frac{d}{dx} x \frac{d}{dx}\right] y^{(2)} = -x^2 y^{(0)}$$

<u>4th</u> $\quad \left[x \frac{d}{dx} x \frac{d}{dx}\right] y^{(4)} = -x^2 y^{(2)}$

$\vdots$

Formally we can solve these equations by constructing a Green fcn which vanishes for $x_0 > x$

$$\left[x \frac{d}{dx} x \frac{d}{dx}\right] G(x, x_0) = \delta(x - x_0)$$

Then

$$y^{(2)}(x) = \int_0^x -x_0^2 y^{(0)}(x_0)\, G(x, x_0)\, dx_0$$

and so on

$$y^{(2n+2)} = \int_0^x -x_0^2\, y^{(2n)}\, G(x,x_0)$$

Then lets do it: The two solutions to the DEQ are $y_1 = 1$ and $y_2 = \ln x$

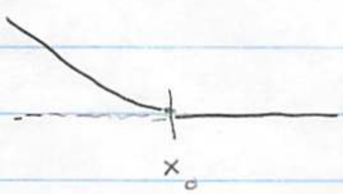
$$G(x) = C\left[ y_1(x)\, y_2(x_0) - y_2(x)\, y_2(x_0) \right] \theta(x - x_0)$$

Integrating across the $\delta$-fcn we have

$$\cancel{x \frac{dG}{dx}\Big|_{x=x_0+\varepsilon}}^{\,0} - x\frac{dG}{dx}\Big|_{x=x_0-\varepsilon} = 1$$

gives $C = 1/x_0$

Picture:



Find $G(x,x_0) = -\ln(x/x_0)/x_0\ \theta(x - x_0)$.

Then

- $$\int_0^x x_0^s\ G(x,x_0) = -\frac{x^s}{s^2}$$

- $$\int x_0^s \ln x\ G(x,x_0) = \frac{\partial}{\partial s}\int_0^x x_0^s\, G(x,x_0)\, dx_0 = \frac{\partial}{\partial s}\left(\frac{-x^s}{s^2}\right)$$

$$= -\frac{x^s}{s^2}\ln x + \frac{2x^s}{s^3}$$

Now just work

$$y^{(2)} = \int_0^x -x_0^2\, y^{(0)}(x_0)\, G(x, x_0)\, dx_0$$

$$y^{(4)} = \int_0^x -x_0^2\, y^{(2)}(x_0)\, G(x, x_0)\, dx_0$$

$$y^{(6)} = \int_0^x -x_0^2\, y^{4}(x_0)\, G(x, x_0)\, dx_0$$

$$\vdots$$

- Starting from $y_1^{(0)} = C_1$ or $y_2^{(0)} = C_2 \ln x$ find

$$y_1(x) = C_1\left[1 - \frac{x^2}{4} + \frac{(x^2/4)^2}{(2!)^2} + \cdots\cdots\right]$$

$$y_2(x) = C_2 \ln x \left[1 - \frac{x^2}{4} + \frac{(x^2/4)^2}{(2!)^2} + \cdots\cdots\right]$$

$$+ \frac{(x^2/4)}{(1!)^2} - (1 + 1/2)\frac{(x^2/4)^2}{(2!)^2} + \cdots\cdots$$

- We see the structure of Fuchs theorem emerge:

$$y_2(x) = C_2\, y_1(x) \ln x \quad + C_2\, x^{s_2} \text{ (power series)}$$

$s_2 = 0$ in this case

(Eq 48.1) $\quad + C_3\, y_1(x)$

can adjust $C_3$ at will

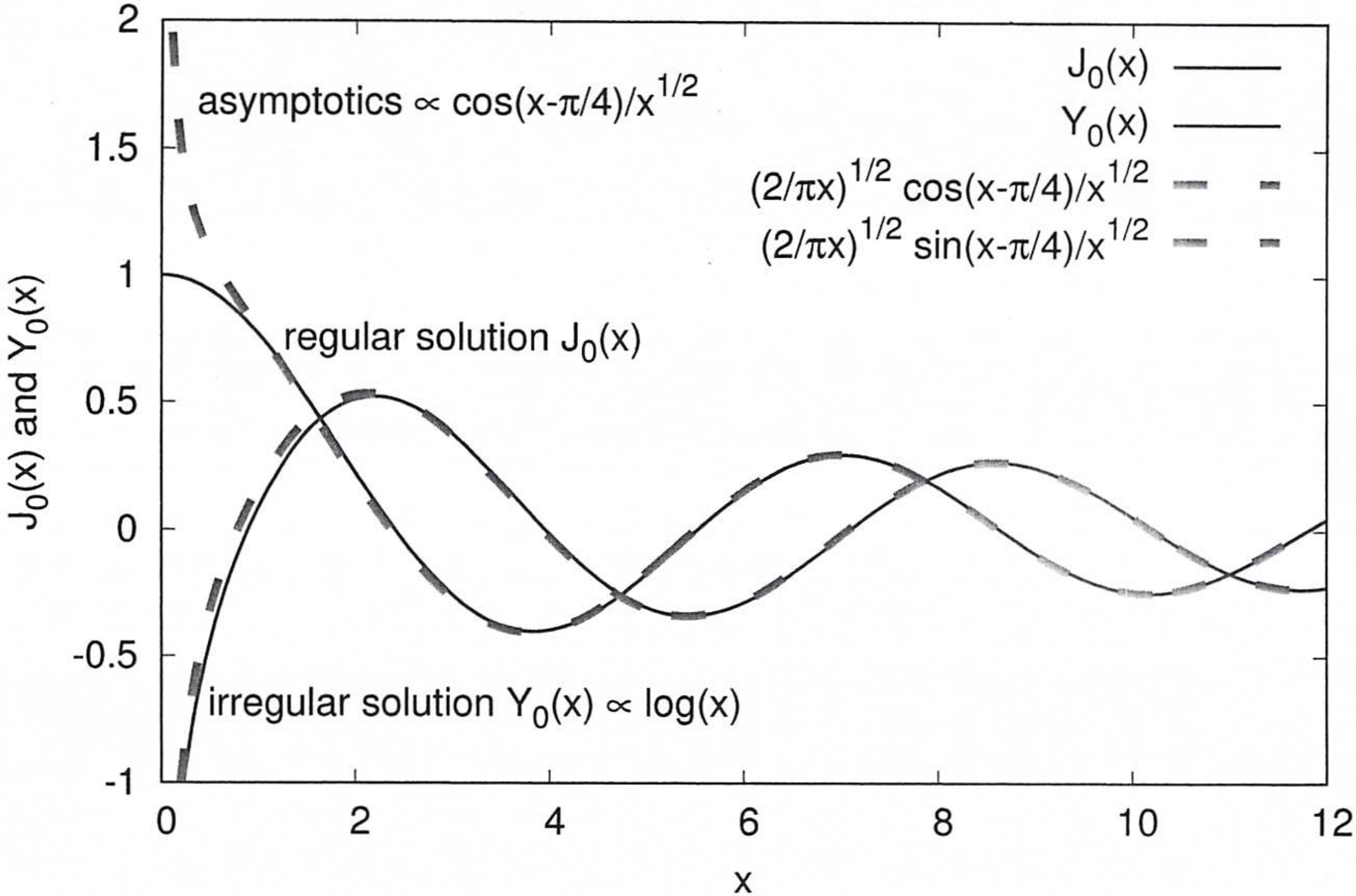
- Then the conventional solutions to the Bessel Eqn choose constants $C_1 = 1$ $C_2 = 2/\pi$ $C_3 = (\gamma_E - \ln 2)\, 2/\pi$

$$y_1(x) = J_\nu(x)\Big|_{\nu=0}$$

$$y_2(x) = \frac{2}{\pi}\frac{\partial J_\nu(x)}{\partial \nu}\Bigg|_{\nu=0} = \frac{2}{\pi}\left(\ln\frac{x}{2} + \gamma_E\right) J_0(x)$$

$$+ \text{ power series}$$

- The coefficients $C_2$ and $C_3$ are adjusted so that $y_2(x)$ has certain behavior as $x \to \infty$. We will discuss this next



$J_0(x)$
$Y_0(x)$
$(2/\pi x)^{1/2}\cos(x-\pi/4)/x^{1/2}$
$(2/\pi x)^{1/2}\sin(x-\pi/4)/x^{1/2}$
asymptotics $\propto \cos(x-\pi/4)/x^{1/2}$
regular solution $J_0(x)$
irregular solution $Y_0(x) \propto \log(x)$
$J_0(x)$ and $Y_0(x)$
x
2
1.5
1
0.5
0
-0.5
-1
0
2
4
6
8
10
12

## Behavior Near an Irregular Sing Point

- For definiteness take Bessel Eqn again $\nu = 0$

$$\left[ x \frac{d}{dx} x \frac{d}{dx} + x^2 - \nu^2 \right] y = 0$$

The irregular singlular point is as $x \to \infty$
Then

$$\left( \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{\nu^2}{x^2} \right) y = 0$$

- Substitute $\boxed{y = e^S}$ $\leftarrow$ This is k general method / substitution

$$\frac{d^2}{dx^2} e^S = e^S \left[ S'' + (S')^2 \right] \simeq e^S \left[ (S')^2 \right]$$

$$\frac{d}{dx} e^S = e^S \left[ S' \right]$$

We can verify a-posteriori that $S'' \ll S$ as $x \to \infty$

$$e^S \left[ (S')^2 + \frac{S'}{x} + 1 \right] = 0$$

- $$S'_{\pm} = -\frac{1}{2x} \pm \frac{\sqrt{1 - 4x^2}}{2x} \underset{x \to \infty}{\Longrightarrow} S'_{\pm}(x) = \pm i - \frac{1}{2x}$$

Then integrating we have

$$S_{\pm}(x) = \pm i x - \frac{1}{2}\log x$$

And thus as $x \to \infty$ our two solutions to the Bessel eqn takes the form

$$y = C_1 e^{S_+} + C_2 e^{S_-}$$

$$y = C_1 \frac{e^{ix}}{\sqrt{x}} + C_2 \frac{e^{-ix}}{\sqrt{x}}$$

or

$$y = C_1 \frac{\cos(x)}{\sqrt{x}} + C_2 \frac{\sin(x)}{\sqrt{x}}$$

Now a more refined approach would work out the corrections to this assymptotic behaviour

$$y_1(x) = \frac{\cos(x)}{\sqrt{x}}\left[1 + \frac{a_1}{x^2} + \frac{a_2}{x^4} + \dots\right]$$

$$y_2(x) = \frac{\sin(x)}{\sqrt{x}}\left[1 + \frac{b_1}{x^2} + \frac{b_2}{x^4} + \dots\right]$$

Our two solutions near the origin asymptote

$$J_0(x) \xrightarrow[x\to\infty]{} \frac{C_1\cos(x)}{\sqrt{x}} + C_2\frac{\sin(x)}{\sqrt{x}}$$

- A global analysis of the solution (which we will not do) establishes (by connecting the perturbative solution at $x \simeq 0$ to the perturbative solution as $x \to \infty$) that

$$J_0(x) \xrightarrow[x\to\infty]{} \sqrt{2/\pi x}\ \cos(z - \pi/4)$$

$$Y_0(x) \xrightarrow[x\to\infty]{} \sqrt{\frac{2}{\pi x}}\ \sin(z - \pi/4)$$

- The normalization of $Y_0(x)$ and how much (i.e. $C_3$) $J_0(x)$ were chosen so that as $x \to \infty$ the two solutions would be precisely out of phase and with the same amplitude.

See Eq (48.1) for the definition of $C_3$