Numerical Approximation

$x+2h$	$x_{-}h$	x_0	$x+h$	$x+2h$
y_{-2}	y_{-1}	y_0	y_1	y_2

$$y_1 = y(x+h) = y_0 + y_0' h + \frac{1}{2} y_0'' h^2$$

$$y_{-1} = y(x-h) = y_0 - y_0' h + \frac{1}{2} y_0'' h^2$$

$$y_2 = y(x+2h) = y_0 + y_0' 2h + \frac{1}{2} y_0'' (2h)^2$$

$$y_{-2} = y(x-2h) = y_0 - y_0' 2h + \frac{1}{2} y_0'' (2h)$$

Approximation

$$y_0' = \frac{y_1 - y_0}{h} + O(h)$$ forward

$$y_0' = \frac{y_0 - y_{-1}}{h} + O(h)$$ backward

$$y_0' = \frac{y_1 - y_{-1}}{2h} + O(h^2)$$ Symmetric

$$y_0'' = \frac{y_1 - 2y_0 + y_{-1}}{h^2} + O(h^2)$$ Symmetric

$$y_0'' = \frac{y_0 - 2y_{-1} + y_{-2}}{h} + O(h)$$ backward

Differential Operators

- Given a vector space on the interval $x \in (a, b)$ with inner product

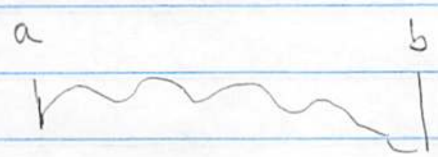
$$\langle f, g \rangle = \int_a^b dx \; w(x) \, f^*(x) \, g(x)$$

- A linear differential operator maps linearly a function in the space to another function in the space. This map (derivative) requires homogeneous b.c.

Example

$$\mathcal{L} = \frac{d}{dt} \qquad \text{with} \qquad \text{b.c.} \qquad V(a) = 0$$

Discretize t between (a, b)



Discretize: $t_n = a + n\overset{\swarrow \Delta t}{h}$ $\qquad t_N = b - h \qquad t_0 = a$

$t_{N+1} = b$

$$\underbrace{t_0 \equiv a \quad t_1 \quad t_2 \quad t_2 \quad t_3 \quad \cdots \quad t_N \quad b}_{}$$

$\leftarrow$ Notation!

$$\frac{dV}{dt} = \frac{V(t_n) - V(t_n - h)}{h}$$

$$\mathcal{L}_t V = \frac{1}{h}\begin{bmatrix} 1 & 0 & & & & \\ -1 & 1 & 0 & & & \\ & -1 & 1 & 0 & & \\ & & -1 & 1 & & \\ & & & & \ddots & 0 \\ & & & & -1 & 1 \end{bmatrix}\begin{bmatrix} V(t_1) \\ V(t_2) \\ \vdots \\ V(t_n) \end{bmatrix}$$

We used the boundary conditions to place the entries in the matrix.

$$\left.\frac{dV}{dt}\right|_{t_1} \equiv \frac{V(t_1) - \cancel{V(t_0)}^{\,0}}{\Delta t} = \frac{V(t_1)}{\Delta t}$$

Moral: A differential operator matrix must specify the b.c

The adjoint

- Given a differential operator $\mathcal{L}_x$, there is a unique differential op $\mathcal{L}_x^\dagger$ (with different b.c.) such that

$$\langle \mathcal{L}_x f, g \rangle = \langle f, \mathcal{L}_x^\dagger g \rangle$$

- i.e.

$$\int_a^b dx\, w(x)\, (\mathcal{L}_x f)^* g = \int_a^b w(x)\, f^*(x)\, (\mathcal{L}_x^\dagger g)$$

We don't need to discretize:

$$\int_a^b dt \, \frac{df}{dt} \, g = \langle \mathcal{L}_t f, g \rangle \qquad \text{where } f(a) = 0$$

Then integrate by parts moving d/dt onto g

$$\int_a^b dt \, \frac{df}{dt} \, g = f(t) \, g(t) \Big|_a^b + \int_a^b dt \, f(t) \left(-\frac{dg}{dt} \right)$$

$$= f(b) \, g(b) + \int_a^b dt \, f(t) \left(-\frac{dg}{dt} \right)$$

To make the boundary terms vanish we choose $g(b) = 0$ then

$$\int_a^b dt \, \frac{df}{dt} \, g = \int_a^b f \left(-\frac{dg}{dt} \right)$$

So

$$\mathcal{L}_t^{\dagger} = -\frac{d}{dt} \qquad \text{with b.c.} \quad g(b) = 0$$

Example: $t \in (a,b)$ with inner product and real fcns

$$\int_a^b dt\; f(t)\, g(t) \simeq \sum_{i=1}^{N} f(t_i)\, g(t_i)$$

- We know from the theory of matrices, that the adjoint is just the hermitian conjugate, i.e

If

$$\mathcal{L}_t = \frac{1}{h}\begin{bmatrix} 1 & 0 & & & \\ -1 & 1 & 0 & & \\ & -1 & 1 & 0 & \\ & & \cdot & \cdot & 0 \\ & & & \cdot & \cdot \end{bmatrix}$$

← zero above diagonal

$\mathcal{L}_t = \frac{d}{dt}$ with $\underbrace{V(a) = 0}_{\text{retarded b.c.}}$

then

$$\mathcal{L}_t^\dagger = \frac{1}{h}\begin{bmatrix} 1 & -1 & & & & \\ 0 & 1 & -1 & & & \\ & 0 & 1 & -1 & & \\ & & 0 & 1 & \cdot & \\ & & & \cdot & \cdot & \\ & & & & 1 & -1 \\ & & & & 0 & 1 \end{bmatrix}$$

zero below diagonal

- So looking at $\mathcal{L}^\dagger V$ we see

$\mathcal{L}^\dagger = -\frac{d}{dt}$ with b.c. $\overbrace{V(b) = 0}^{\text{advanced b.c.}}$

since

$$-\frac{dV}{dt}\bigg|_{t_N} = \frac{-(V(t_{N+1}) - V(t_N))}{h} = \frac{V(t_N)}{h}$$

with $V(t_{N+1}) \to 0$

Example 2

Harmonic oscillator with retarded b. c.

$$\mathcal{L}_t = \left[\frac{d^2}{dt^2} + \omega_0^2 \right] \qquad x(a) = 0 \qquad \frac{dx(a)}{dt} = 0$$

Then

$$\int_a^b dt \left[\left(\frac{d^2}{dt^2} + \omega_0^2 \right) x(t) \right] y(t)$$

Again we integrate by parts (twice) moving d^2x/dt^2 to d^2y/dt^2

$$\int_a^b dt \frac{d^2x}{dt} y = y \frac{dx}{dt} \Big|_a^b - \int_a^b \frac{dx}{dt} \frac{dy}{dt}$$

$$= \left(y \frac{dx}{dt} - x \frac{dy}{dt} \right) \Big|_a^b + \int_a^b dt \, x \frac{d^2y}{dt^2}$$

$$= y(b) \frac{dx}{dt}(b) - x(b) \frac{dy}{d\tau}(b) + \int_a^b dt \, x \frac{d^2y}{dt^2}$$

Requiring the surface terms to vanish we have the boundary conditions

$$y(b) = y'(b) = 0$$

Thus

$$\mathcal{L}_t = \frac{d^2}{dt^2} + \omega_0^2 \qquad \underbrace{x(a) = \dot{x}(a) = 0}_{\text{retarded b.c.}}$$

$$\mathcal{L}_t^\dagger = \frac{d^2}{dt^2} + \omega_0^2 \qquad \underbrace{x(b) = \dot{x}(b) = 0}_{\text{advanced b.c}}$$

- although these look the same. They are not the same since they obey different b.c.
- We say that the two operators are only formally self adjoint.

$\frac{d^2}{dt^2}$ with $x(a) = \dot{x}(a) = 0$ is discretized as

$$\frac{d^2}{dt^2} = \frac{1}{h^2}\begin{bmatrix} 1 & 0 & 0 & & \\ -2 & 1 & 0 & & \\ 1 & -2 & 1 & 0 & \\ & 1 & -2 & 1 & \\ & & \cdot & \cdot & \cdot \\ & & & & \cdot \end{bmatrix}$$

diagonal entries = 1

which is not a symmetric matrix

Example 3

$$\mathcal{L} = \frac{d}{dx^2} + k^2 \qquad \text{with b.c. } y(a) = y(b) = 0$$

- Then $\mathcal{L}^{\dagger} = \mathcal{L}$. Proof, Follow the previous example

$$\int_a^b dx\, f(x)\left(\mathcal{L}\, g(x)\right) = f(x)g'(x) - f'(x)g(x)\Big|_a^b + \int_a^b dx\left(\left[+\frac{d^2}{dx^2} + k^2\right] f(x)\right) g(x)$$

$$= \int_a^b dx \left[\left(\frac{\partial^2}{\partial x^2} + k^2\right) f(x)\right] g(x)$$

$$= \int_a^b dx\, \left(\mathcal{L}\, f(x)\right) g(x)$$

- The discretization is

$$\frac{d^2}{dx^2} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & \\ & & & \ddots \end{pmatrix}$$

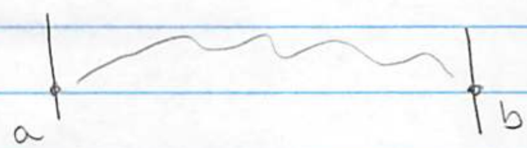
diagonal entries $= -2$

this is symmetric

Example 4 - Test yoursleif

$$\mathcal{L}_t = m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2 \qquad x(a) = \dot{x}(a) = 0$$

$$\mathcal{L}_t^{\dagger} = m\frac{d^2}{dt^2} - m\eta\frac{d}{dt} + m\omega_0^2 \qquad x(b) = \dot{x}(b) = 0$$



Ex 5

time dependent mass

$$\mathcal{L}_t = m(t)\frac{d}{dt} + \eta \qquad y(a) = 0$$

$$\mathcal{L}_t^{\dagger} = \left[\frac{d}{dt}\big(m(t) + \eta\right] \qquad y(b) = 0$$

The Green Reciprocity Theorem

green function

- So far considered $G(x, x_0)$ to be a function of x at fixed x_0. Can also determine it as a function of x_0.

- It turns out that if

$$\mathcal{L}_x G(x, x_0) = \delta(x - x_0) \quad \text{with boundary conditions}$$

Then with respect to x_0, $G(x, x_0)$ obeys the adjoint differential equation with adjoint b.c.

$$\mathcal{L}^{\dagger}_{x_0} G(x, x_0) = \delta(x - x_0) \quad \text{with adjoint b.c}$$

Example

$$\left(\frac{d}{dt} + \eta\right) G(t, t_0) = \delta(t - t_0) \qquad \text{with } G(t, t_0) = 0 \text{ for } t < t_0$$

Recall $G(t, t_0) = \Theta(t - t_0)\, e^{-\eta(t - t_0)}$. The theorem says

$$\underbrace{\left(-\frac{d}{dt_0} + \eta\right)}_{\text{adjoint op.}} G(t, t_0) = \delta(t - t_0) \qquad \underbrace{\text{with } G(t, t_0) = 0 \text{ for } t_0 > t}_{\text{adjoint b.c.}}$$

For this problem, note $G(t, t_0) = f(t - t_0)$ so the t_0 dependence is clear, $d/dt \rightarrow -d/dt_0$

Proof

$$\mathcal{L}_x G(x, x_0) = \delta(x - x_0)$$

- So for all x, x_0 we have

$$G(x, x_0) = \int_a^b dx' \, G(x, x') \, \mathcal{L}_{x'} G(x', x_0)$$

Now as in our examples (e.g. $\mathcal{L}_{x'} = \frac{d^2}{dx'^2}$) we integrate by parts to move the operator onto $G(x, x')$

$$G(x, x_0) = \int_a^b dx' (\mathcal{L}_{x'}^{\dagger} G(x, x')) \, G(x', x_0) + \text{bndry terms}$$

This has to hold for all x, x_0 (a 2d number conditions) ∴ the only way to satisfy these conditions is if

$$\mathcal{L}_{x'}^{\dagger} G(x, x') = \delta(x - x')$$

and require that the boundary terms vanish. Requiring that the boundary terms vanish means that $G(x, x')$ satisfies adjoint b.c. with respect to x'

Green Thrm Overview

- The Green function can be used to construct a formal solution to any linear problem.

- For instance in Quantum Mechanics

$$\Psi(x,t) = \int dx_0 \underbrace{G(t\,x \mid t_0\,x_0)}_{\text{propagator}} \underbrace{\Psi(t_0\,x_0)}_{\text{initial condition}}$$

- Let us show how this generalizes for any linear problem

Example: want to solve $t \in [a,b)$

$$\frac{dV}{dt} + \eta V = f(t) \qquad V(a) = V_a$$

initial cond

$$\mathcal{L}_t = \frac{d}{dt} + \eta \qquad \text{with} \quad V(a) = 0$$

homogeneous retarded b.c.

$$\mathcal{L}^\dagger_{t_0} = -\frac{d}{dt} + \eta \qquad \text{with} \quad V(b) = 0$$

the adjoint b.c.

Then the solution is $V(t)$

$$V(t) = \int_a^b dt_0 \; V(t_0)\, \mathcal{L}^\dagger_{t_0} G(t, t_0)$$

see previous section

$$\mathcal{L}^\dagger_{t_0} G(t,t_0) = \delta(t-t_0)$$

Now

$$V(t) = \int_a^b dt_0 \; V(t_0) \left(-\frac{d}{dt_0} + \eta \right) G(t, t_0)$$

by parts

$$= \int_a^b dt_0 \left(\frac{d}{dt_0} + \eta \right) V(t_0) \; G(t, t_0)$$

$$+ \left(-V(t_0) G(t, t_0) \right) \Big|_{t_0 = a}^{t_0 = b}$$

$$= \int_a^b f(t) \, G(t, t_0) \quad + \quad -V(b) \overset{0}{\cancel{G(t,b)}} + V(a) \, G(t,a)$$

by adjoint b.c.

$$V(t) = \underbrace{\int_a^b f(t) \, G(t, t_0)}_{\text{particular solution}} + \underbrace{V(a) \, G(t, a)}_{\text{homogeneous solution}}$$

- Thus we have expressed the homogeneous solution as the "convolution" of the initial condition with the Green function. The same procedure works for all linear equations

Example

- Harmonic Oscillator. Want to solve $t \in [a,b]$

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right) x = 0 \qquad \text{with i.c.} \quad x(a) = x(a)$$

$$\left.\frac{dx}{dt}\right|_{t=a} = \partial_{t_a} x(a)$$

- Then lets call the solution $x(t)$

$$\mathcal{L} \equiv \frac{d^2}{dt^2} + \omega_0^2 \qquad \text{with retarded b.c}$$

$$\mathcal{L}^{+} \equiv \frac{d}{dt_0^2} + \omega_0^2 \qquad \text{with advanced b.c.}$$

$$\mathcal{L}^{+}_{t_0} G = \left(\frac{d^2}{dt_0^2} + \omega_0^2\right) G(t,t_0) = \delta(t-t_0) \qquad G(t,b) = 0$$

$$\left.\frac{\partial G(t,t_0)}{\partial t_0}\right|_{t_0=b} = 0$$

Then $x(t)$ is

$$x(t) = \int_a^b x(t_0) \overbrace{\left(\frac{d^2}{dt_0^2} + \omega_0^2\right) G(t,t_0)}^{=\delta(t-t_0)} \, dt_0$$

Integrate twice by parts, use $(d^2/dt^2 + \omega_0^2)x = 0$

$$x(t) = x(t_0)\,\partial_{t_0} G(t,t_0) - \partial_{t_0} x(t_0)\, G(t,t_0) \Big|_a^b$$

$$\boxed{x(t) = x(a)\,\partial_{t_a} G(t,t_a) - \partial_{t_a} x \; G(t,t_a)}$$

Summary: the solution is the Wronskian of the initial conditions and the Green function for SHO

Compare first order equations $G \equiv G(tx \mid t_0 x_0)$

$$\frac{dV}{dt} + \eta V = 0 \qquad V(t) = G(t,a)\, V(a)$$

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} H \psi \qquad \psi(t,x) = \int_{x_0} G(tx \mid t_0 x_0)\, \psi(x_0)$$

to second order equations $G \equiv G(tx \mid t_0 x_0)$

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right) x = 0 \qquad x(t) = x_0\, \partial_{t_0} G(t,t_0) - \partial_{t_0} x\, G(t,t_0)$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) u = 0 \qquad u(t,x) = \int dx_0 \left[u(t_0 x_0)\, \partial_{t_0} G - \partial_{t_0} u(t_0 x_0)\, G \right]$$

We will treat the Schrödinger and wave equations more systematically later. For now just understand the analogy

Self Adjoint Linear Ops and Eigenvalue Problems

- Take the inner product

$$\langle f, g \rangle = \int_a^b w(x)\, f^*(x)\, g(x)$$

- With the right b.c. the Sturm-Liouiville Op is self adjoint $w(x) > 0$ $p(x), q(x)$ real

$$\mathcal{L}_x \equiv \frac{1}{w(x)} \left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right]$$

- Just integrate twice by parts (Do it!)

$$\int_a^b w(x) (\mathcal{L}_x \phi_1)^* \phi_2 = -p(x) \left(\frac{d\phi_1^*}{dx} \phi_2 - \phi_1^* \frac{d\phi_2}{dx} \right) \Bigg|_a^b$$

$$+ \int_a^b dx\, w(x)\, \phi_1^*(x) (\mathcal{L}_x \phi_2)$$

So the operator is self adjoint if

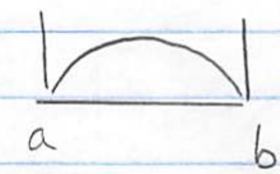
$$\underbrace{-p(x) \left(\frac{d\phi_1^*}{dx} \phi_2 - \phi_1^* \frac{d\phi_2}{dx} \right)}_{\text{Bndry Terms}} \Bigg|_a^b = 0$$

Bndry Terms ← this is known as the bilinear concomitant, i.e. flux

Examples of B.C. Where Boundary Terms Vanish

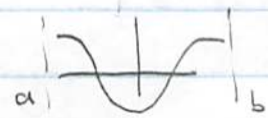
- Particle in box, Dirichlet B.C.

$$\left(\frac{d^2}{dx^2} + k^2\right)\phi = 0 \qquad \phi(a) = \phi(b) = 0$$



With these b.c.
the boundary terms vanish
a the operator is self adjoint

- Neumann b.c. $\phi'(a) = \phi'(b) = 0$ also
leads to a self-adjoint op.



- In general a two-point
homogeneous b.c. of the form

$$\alpha\phi + \beta\phi' = 0 \quad \text{at} \quad x = a \ \underline{\text{and}} \ x = b$$

leads to a self-adjoint op.

- Regularity at singular points.
Often a or b will correspond to
zero's of $p(x)$ (which is a singular point
of the DEQ). Demanding regularity at
this point, we have for instance

$$\lim_{x \to a} p(x)\left(\frac{d\phi_1^*}{dx}\phi_2 - \phi_1^*\frac{d\phi_2}{dx}\right) = 0 \qquad p(a) = 0$$

ϕ regular

Eigen-fcns of Sturm Liouville operators

- We Look for solutions of the form

$$\frac{1}{w(x)}\left[\frac{-d}{dx} p(x) \frac{d}{dx} + q(x)\right]\phi_n = \lambda_n \phi_n$$

or

$$\left[\frac{-d}{dx} p(x) \frac{d}{dx} + q(x)\right]\phi_n = \lambda_n w(x)\, \phi_n$$

with self-adjoint b.c.

The eigen functions are orthogonal:

① $$\langle \phi_{n_1}, \phi_{n_2}\rangle = \int_a^b dx\; w(x)\, \phi^*_{n_1} \phi_{n_2} = \begin{cases} 0 & \lambda_1 \neq \lambda_2 \\ C_n & \lambda_1 = \lambda_2 \end{cases}$$

And complete

② $$\sum_n \frac{\phi_n(x)\, \phi^*_n(x')}{C_n} = \frac{1}{w(x)}\, \delta(x-x')$$

Any function satisfying the b.c. can be expanded

- $$f(x) = \sum_\lambda f_n \frac{\phi_n}{C_n}$$

- $$f_n = \int_a^b dx\; w(x)\, \phi^*_n(x)\, f(x) = \langle \phi_n, f\rangle$$

Consistency of ②

$$f(x) = \sum_n \underbrace{\int_a^b dx'\, w(x')\, \phi_n^*(x')\, f(x')}_{f_n} \frac{\phi_n(x)}{c_n}$$

$$= \int_a^b dx'\, w(x') \sum_n \frac{\phi_n(x)\, \phi_n^*(x')}{C_n} f(x')$$

$$= \int_a^b dx'\, w(x') \frac{1}{W(x')} \delta(x-x')\, f(x') = f(x)$$

- Then Proof of ①. For <u>any</u> two functions ϕ_1 and ϕ_2 we showed

$$\int_a^b dx\, w(x) \left(\mathcal{L}_x \phi_1\right)^* \phi_2 = -p(x) \left(\frac{d\phi_1^*}{dx}\phi_2 - \phi_1^* \frac{d\phi_2}{dx}\right)\Bigg|_a^b + \int_a^b dx\, w(x)\, \phi_1^* (\mathcal{L}_x \phi_2)$$

- Now if ϕ_1 and ϕ_2 are eigen-fcns obeying the b.c. then the boundary terms vanish. And $\mathcal{L}_x \phi_n = \lambda_n \phi_n$ yielding (note $\mathcal{L}_x(\phi^*) = (\mathcal{L}_x \phi)^* = \lambda\, \phi^*$

$$(\lambda_1 - \lambda_2) \int_a^b dx\, w(x)\, \phi_1^*(x)\, \phi_2(x) = 0$$

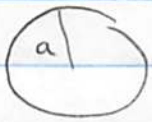
So provided $\lambda_1 \neq \lambda_2$ we have

$$\langle \phi_1, \phi_2 \rangle = 0$$

Particle in a Circle of radius a

$$-\nabla^2 \psi = k^2 \psi \qquad \frac{\hbar^2 k^2}{2m} = E_k$$

ψ vanishes on boundary $\rho = a$
and is regular in interior



- Separate variables $\psi = R(\rho)\,\Phi(\phi)$

$$-\nabla^2 = -\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{-\partial^2}{\partial \phi^2}$$

- Look at $-\nabla^2\psi/\psi \times \rho^2$

$$\underbrace{\frac{1}{R\,\cancel{\Phi}}\,-\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R(\rho)\,\cancel{\Phi}}_{\text{if } \phi \text{ changes this is constant}} + \underbrace{\frac{1}{\cancel{R}\,\Phi}\,\frac{-\partial^2 \cancel{R}\phi}{\partial\phi^2}}_{\text{if } \rho \text{ changes this is constant call it } m^2} = k^2\rho^2$$

① $$-\frac{\partial^2 \Phi}{\partial \phi^2} = m^2\,\Phi$$

② $$-\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R(\rho) + \left(m^2 - (k\rho)^2\right)R = 0$$

- The field Φ is periodic (this is a self adjoint and homogeneous b.c.)

$$\Phi_m = e^{im\phi} \qquad m = \text{integer}$$

$$\Phi_m \Big|_{2\pi} = \Phi_m \Big|_0 \quad \text{leads to} \quad m = \text{integer}$$

- Lets look at the second equation. Define $\underline{\underline{x = k\rho}}$
  $R(x) = R(\rho)$

$$-x \frac{\partial}{\partial x} x \frac{\partial}{\partial x} R(x) + (m^2 - x^2) R(x) = 0$$

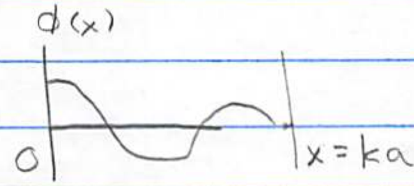
this is $p(x)$ it vanishes at $x = 0$

This is Bessel Eqn. $x = 0$ is a reg-sing point
Take for example $m = 0$ for example

$$R(x) = C_1 + \underbrace{C_2 \ln x}_{\text{irregular as } x \to 0}$$

- So we have two self adjoint b.c. on $R(x)$. The requirement of regularity as $x \to 0$

$$x \left(\frac{d\phi_1^*}{dx} \phi_2 - \phi_1^* \frac{d\phi_2}{dx} \right) \Bigg|_a^b = 0$$



and that $\phi(ka) = 0$. With these requirements the bilinear concomitant vanishes and the operator is self adjoint. It is easy to see that

- the bilinear form would not vanish if we allowed logs $\left. x \frac{\partial}{\partial x} \log x \right|_{x \to 0} = 1$

- So the quantization procedes as follows. Take $m = 0$ for definiteness. The general solution to the DEQ is

$$R(x) = \underbrace{C_1 J_0(x)}_{\text{regular}} + \underbrace{C_2 Y_0(x)}_{\text{irregular}}$$

demand regularity

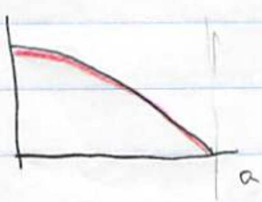
$$R(\rho) = J_0(k\rho)$$

- In passing to the second line we discarded the irregular solution and chose the normalization $C_1 = 1$. Now we have the requirement that

$$\left. J_0(k\rho) \right|_{\rho = a} = 0 \qquad \text{i.e.} \quad k_n a = \text{zeros of the } J_0 \text{ bessel fcn} \equiv x_{0n}$$

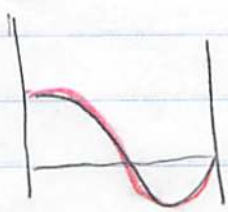
$$x_{0n} = 2.4,\ 5.52,\ 8.65 \ldots\ldots, \quad n = 1, 2, 3 \ldots$$

<u>Picture</u>

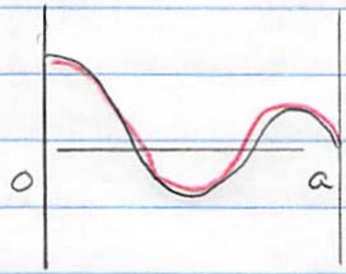


$$k_1 a = 2.404 = x_{01} \longleftarrow \quad E_n = \frac{\hbar^2 k_n^2}{2m}$$

quantized energy



$$k_2 a = 5.52008 = x_{02}$$



$k_3 a = 8.65 = x_{03}$

For large n and $x \to \infty$

equally spaced with J_ν and $J_{\nu+1}$ out of phase

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad x_{\nu n} \simeq n\pi + \left(\nu - \frac{1}{2}\right)\frac{\pi}{2}$$

Summary this is a Sturm Liouvill e-value problem
The eigen function and evalues are

$$\phi_n(\rho) = J_0(k_n\rho) \qquad \text{where} \quad k_n a = x_{0n}$$

and the satisfy y

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\phi_n(\rho) = k_n^2\,\phi_n$$

So $w(\rho) = \rho \quad \underline{p}(\rho) = \rho \quad q(\rho) = 0.$

- The orthogonality reads for this oprerator

$$\int_0^a \rho\, J_0(k_n\rho)\, J_0(k_m\rho) = C_n\,\delta_{nm}$$

- Completeness says

$$\sum_n \frac{J_0(k_n\rho)\, J_0(k_n\rho')}{C_n} = \frac{1}{\rho}\,\delta(\rho-\rho')$$

weight

- Using the differential equation, and the recurrence relation:

$$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x} J_\nu$$

Then we can show (easy but not shown)

$$\int_0^a \rho\, J_0(k_n\rho)\, J_0(k_n\rho) \equiv C_n = \frac{a^2}{2}\left[J_1(k_n a)\right]^2$$

Leading to the Fourier-Bessel Series

(1) $$f(\rho) = \sum_{n=1}^{\infty} f_n \frac{J_0(k_n\rho)}{C_n}$$

(2) $$f_n = \int_0^a \rho\, f(\rho)\, J_0(k_n\rho)$$

- In the limit that $a \to \infty$ with $k = k_n$ fixed then

$$x_n = k_n a \longrightarrow \infty \qquad \text{and } n \to \infty \qquad \sum_n \to \int dn = \int \frac{dk}{\pi} a$$

The bessel zeros are $n\pi - \pi/4 = k_n a$

Then we have after some analysis (not shown)

$$C_n \to \frac{a^2}{2} \frac{2}{\pi (k_n a)} \qquad \text{use} \qquad J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

Then we have that (1) becomes

$$\boxed{\begin{aligned} f(\rho) &= \int_0^\infty k\,dk\; f(k)\, J_0(k\rho) \\ f(k) &= \int_0^\infty \rho\, d\rho\; f(\rho)\, J_0(k\rho) \end{aligned}}$$

$$\sum_n \frac{1}{C_n} \to \int_0^\infty k\,dk$$

Thus we have rederived the Hankel Transform from our Eigen-expansion.