Equivalent Representations

- Given a matrix representation $D(g)$ we can find an equivalent representation by a similarity transformation

$$\underline{D}(g) = S^{-1} D(g) S \quad \text{or} \quad S^{-1} O_g S = \underline{O}_g$$

Then $\underline{D}(g)$ is also a matrix rep

$$\underline{D}(g_1)\,\underline{D}(g_2) = S^{-1} D(g_1) S S^{-1} D(g_2) S$$

$$= S^{-1} D(g_1) D(g_2) S$$

$$= S^{-1} D(g_1 g_2) S = \underline{D}(g_1 g_2)$$

- The trace of the matrix is known as the "character of the representation" with symbol χ

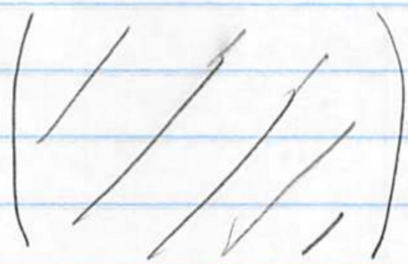
$$\boxed{\chi(g) \equiv \mathrm{Tr}\, D(g)}$$

Note the trace is or charachter is independent of any similarity transform

Reducable Representations

- A representation of the group is completely reducable if through a similarity transformation it can be brought to black diagonal form for all $D(g)$. of the representation

i.e. if

$$D(g) =$$ 

Then by change of basis

$$\underline{D}(g) = S\, D(g)\, S^{-1}$$

The new matrix is block diagonalized

$$\underline{D}(g) = \left(\begin{array}{c|c} D^{(1)} & 0 \\ \hline 0 & D^{(2)} \end{array}\right) \equiv D^{(1)} \oplus D^{(2)}(g)$$

- If a representation can not be block diagonalized it is irreducable

- Finding the irreducable representations, and the decomposition, e.g. $D(g) = D^{(1)}(g) \oplus D^{(2)}(g)$ is our chief task!

- A general reducable rep can be decomposed

$$D(g) = D^{(1)} \oplus D^{(1)} \oplus D^{(2)} \oplus D^{(2)} \oplus D^{(2)} \oplus \cdots$$

$$= 2D^{(1)} \oplus 3D^{(2)} \oplus \cdots \equiv \sum_i a_i D^{(i)}$$

Formal Direct sum of irreducable reps.

means

$$D(g) = \begin{pmatrix} D_1 & & & & \\ & D_1 & & & \\ & & D_2 & & \\ & & & D_2 & \\ & & & & D_2 \end{pmatrix}$$

- The trace of this matrix is

the # of times $D^{(i)}$ appears

$$\chi(g) = \sum_i a_i \chi^{(i)}(g)$$

trace of full matrix

trace of the irreducable reps matrices

- Normally if I have a vector in vector space $\mathbb{V}$

$$|v\rangle = v^a |e_a\rangle$$

Then the action of the group operations mixes the components of the vector

$$v^a \xrightarrow[g]{} D^a{}_b(g)\, v^a$$

- The reduction separates the vector space $\mathbb{V}$ into direct sum $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$. Each vector $|\vec{v}\rangle$ can be written

$$|\vec{v}\rangle = |\vec{v}_1\rangle + |\vec{v}_2\rangle \qquad |v_1\rangle \in \mathbb{V}_1 \text{ and } |v_2\rangle \in \mathbb{V}_2$$

- and the components of $|\vec{v}_1\rangle$ do not mix with $|\vec{v}_2\rangle$ under the ~~with each other~~ under the action of the group (though the components of $|\vec{v}_1\rangle$ mix amongst itsself).

The vector space $\mathbb{V}_1$ and $\mathbb{V}_2$ constitute invariant subspaces, i.e. the action of the group on a vector in $\mathbb{V}_1$ returns a vector in $\mathbb{V}_1$

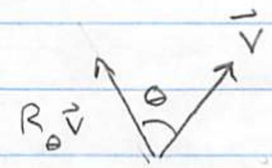
$$v_1^a \xrightarrow[g]{} D^{(1)a}{}_b\, v_1^b$$

$$v_2^{(a)} \xrightarrow[g]{} D^{(2)a}{}_b\, v_2^b$$

Example of a reducable rep. Circular Polorization

- Consider the set of 2x2 matrices representing the rotations around the Z-axis:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = D(R_\theta)$$

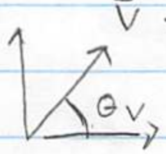


This is a group (an abelian group)

$$R_\theta R_{\theta'} = R_{\theta+\theta'}$$

It is a continuous group (the number of elements is not fixed). Consider a change of basis of $D(R_\theta)$

$$\vec{v} = v_+^* \vec{e}_+ + v_-^* \vec{e}_-$$



$$\vec{e}_\pm \equiv \frac{\vec{e}_x \pm i\vec{e}_y}{\sqrt{2}}$$

$$v_\pm \equiv \frac{v_x \pm i v_y}{\sqrt{2}} = \frac{|\vec{v}|(\cos\theta_v \pm i\sin\theta_v)}{\sqrt{2}} = \frac{|\vec{v}|e^{i\theta_v}}{\sqrt{2}}$$

- So under rotation $\theta_v \rightarrow \theta_v + \theta$ and thus

$$v_+ \xrightarrow[R_\theta]{} \frac{|v|}{\sqrt{2}} e^{i(\theta_v+\theta)} = e^{i\theta} v_+$$

$$v_-^* \xrightarrow[R_\theta]{} \frac{|v|}{\sqrt{2}} e^{i(\theta_v+\theta)} = e^{-i\theta} v_-$$

Thus in our new basis we have the transformation

$$\begin{pmatrix} \underline{V}_+ \\ \underline{V}_- \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

V_+, V_- are known as the spherical components of a vector $\vec{V}$. They have the advantage over V^x, V^y. They transform as an irreducable representation under rotations.

- Under rotation one linearly polarization becomes another
- Howeuer one circulary polarized state just picks up a phase under rotations

Some Theorems Without Proof

(1) The number of inequivalent irreducable representations and there dimensions is very limitted:

Let $D^{(\mu)}(g)$ label the irreps with

$\mu = 1, 2, \ldots N_{reps}$

of dimension n_μ ($D^{(\mu)}$ is a $n_\mu \times n_\mu$ matrix)

Then

$$\boxed{\sum_{\mu=1}^{N_{reps}} n_\mu^2 = n_G}$$

order of group $= 6$ for D_3, 3 for Z_3

• We will later prove that identity, alternating, and matrix reps are the only irreps of D_3. Thus

$n_\mu = 1$ $n_\mu = 1$ $n_\mu = 2$

$$1^2 + 1^2 + 2^2 = 6$$

(2) Schur's Lemma I

Let $D^{(\mu)}(g)$ be an irrep of dimension $n_\mu \times n_\mu$. Let A be a square matrix which commutes with $D^{(\mu)}(g)$ for all $g \in G$

$$[A, D^{(\mu)}(g)] = 0$$

Then either $A = 0$ or $A = \lambda \mathbb{1}$

Proof See Hammermesh (3.14)

3. Schur's Lemma 2

If $D^{(1)}$ and $D^{(2)}$ are two inequivalent reps of dimensions $n_{(1)}$ and $n_{(2)}$ respectively (can be different or same)

Then if a matrix A "intertwines" $D^{(1)}$ and $D^{(2)}$ meaning

$$D^{(1)}(g)\, A = A\, D^{(2)}(g) \qquad \left(D^{(1)}\right)\left(A\right) = \left(A\right)\left(D^{(2)}\right)$$

Then A is zero.

4. Unitarity (for a finite, or compact group)

Any matrix representation is equivalent to a unitary representation

A matrix is unitary if $D^{\dagger} = D^{-1}$ $\quad D^{\dagger} D = \mathbb{1}$

Then a rep is unitary when

$$D_{ab}(g^{-1}) = (D^{-1}(g))_{ab} = (D^{\dagger})_{ab} = (D^{*})_{ba}$$

The Great Orthogonality Theorem Without Proof

- Let the group have irreps $D^{(\mu)}(g)$ with $(\mu) = 1 \ldots N_{reps}$ of dimension $n_\mu \times n_\mu$

- Then the group averaged matrix elements are maximally orthogonal

$$\frac{1}{n_G} \sum_g D^{(\mu)}_{ab}(g)\, D^{(\nu)}_{dc}(g^{-1}) = \frac{I}{n_\mu} \delta_{\mu\nu} \delta_{ac} \delta_{bd}$$

Or since D is equalent to a unitary rep

$$\boxed{\frac{1}{n_G} \sum_g D^{(\mu)}_{ab}(g) \left(D^{(\nu)}_{cd}(g)\right)^* = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ac} \delta_{bd}}$$

- We can phrase this like this. Consider "vectors" in the vector space of group operators. The notation is

$$\hat{V} = \sum_{i=1}^{n_G} V(g_i)\, \hat{g}_i = \sum_i v_i\, \hat{g}_i = \sum_g V(g)\, \hat{g}$$

- We can add and subtract operators and take a "group" inner product

$$\langle \hat{V}, \hat{W} \rangle = \sum_{i=1}^{n_G} v_i^* w_i = \sum_g v^*(g)\, w(g)$$

Then define the "vectors" in this space of operators

$$\boxed{\hat{e}^{(\mu)}_{ab} \equiv \left(\frac{n_\mu}{n_G}\right) \sum_g D^{*(\mu)}_{ab}(g)\, \hat{g}}$$

n_μ / n_G inserted for later convenience

These vectors are orthogonal

$$\boxed{\langle \hat{e}^{(\mu)}_{ab}, \hat{e}^{(\nu)}_{cd} \rangle = \frac{n_\mu}{n_G} \delta_{\mu\nu} \delta_{ac} \delta_{bd}}$$

- Thus we have

$$\sum_{\mu=1}^{N_{reps}} n_\mu^2$$

orthogonal vectors in a "vector space" of dimension n_G we must have

$$\sum_{\mu=1}^{N_{reps}} n_\mu^2 \leq n_G$$

- In fact $\sum n_\mu^2 = n_G$ and thus the vectors

$$\hat{e}^{(\mu)}_{ab}$$

Form a complete orthogonal basis for the group algebra (i.e. the group vector space) !

Functions of Definite Symmetry

- Start with a function f. By acting on f with the group operator we get new functions

$$\overset{f_1}{f}, \overset{f_2}{O_{r_1} f}, \overset{f_3}{O_{r_2} f}, \overset{f_4}{O_{s_0} f}, \overset{f_5}{O_{s_1} f}, \overset{f_6}{O_{s_2} f}$$

Then any function in the span of these functions is represented by the "vector", $\hat{c} = c_i \hat{g}_i$

$$O_{\hat{c}} = \sum_i \underbrace{c_i}_{\text{numbers}} \hat{O}_i = \sum_g c(g) \hat{O}_g$$

$$O_{\hat{c}} f = \sum c_i \hat{O}_i f$$

$$= \sum_i c_i f_i$$ $\leftarrow$ a function in the span

- Instead of using the basis

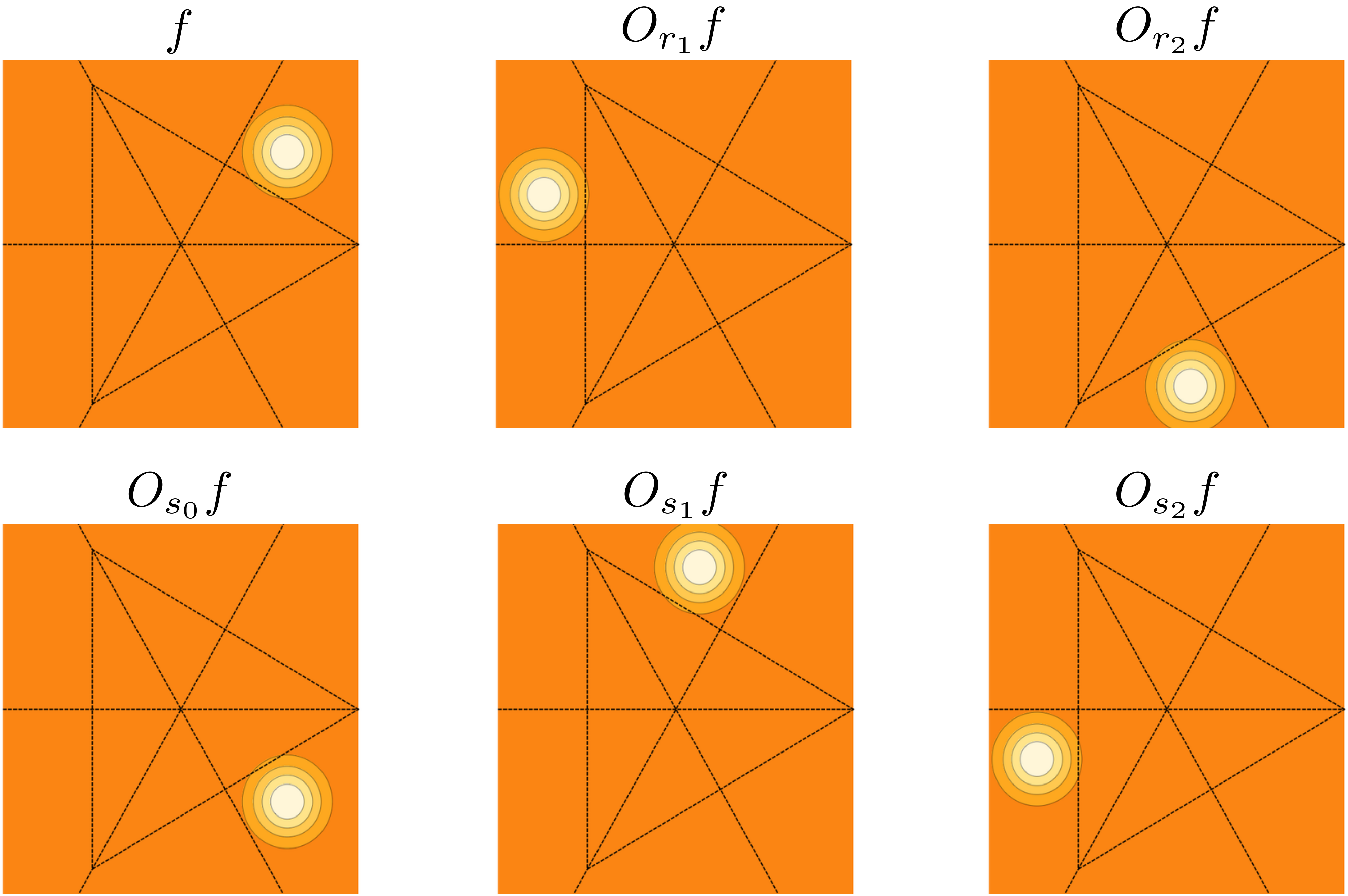
$$f_i \equiv \hat{O}_i f$$

We will use

$$f_{ab}^{(\mu)} = \hat{e}_{ab}^{(\mu)} f = \frac{n_\mu}{n_G} \sum_i (D_{ab}^{(\mu)}(g_i))^* \hat{O}_{g_i} f$$

They have the same dimensionality

The vector space = linear span of six functions



- Take the identity rep $n_\mu = 1$ $D^{(\mu)}_{11}(g) = 1$

$$f^{(1)}_{11} = \frac{1}{n_G} \sum_g O_g f$$

$$f^{(1)}_{11} = \frac{1}{6} \sum_i f_i$$ ← We looked this earlier $f^{(1)}_{11} = f_S$

This is invariant under the group ops

$$O_g f^{(\mu)}_{11} = f^{(\mu)}_{11}$$

- Similarly look at the alternating reps

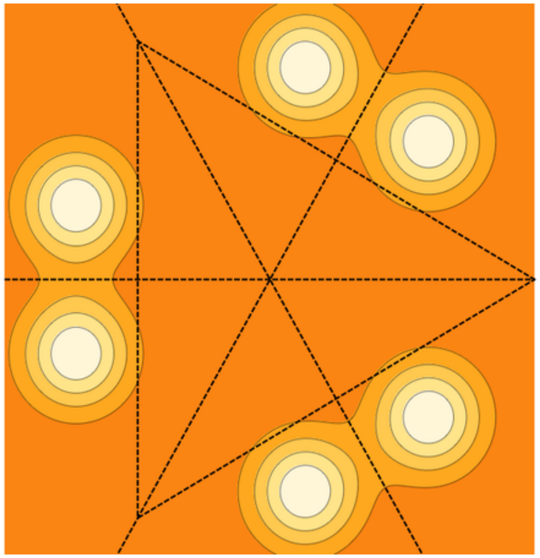
$$f_A = f^{(2)}_{11} = \frac{1}{6}\left(f + O_{r_1} f + O_{r_2} f - O_{s_0} f - O_{s_1} f - O_{s_2} f \right)$$

Then this is, up to a sign, invariant:

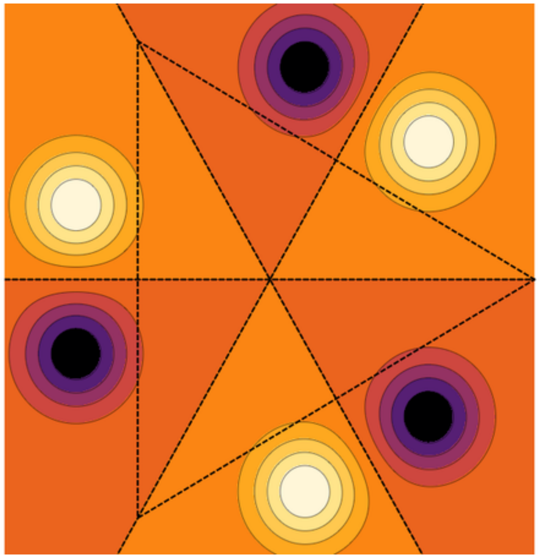
$$O_g f^{(2)}_{11} = \pm f^{(2)}_{11}$$

where + is for $O_{r_0}, O_{r_1}, O_{r_2}$ and - is for $O_{s_0}, O_{s_1}, O_{s_2}$

These are shown on the slides



$$f_S(x) \equiv f_{11}^{(1)}(x)$$



$$f_A(x) \equiv f_{11}^{(2)}$$

- Now look at $f_{11}^{(3)}$ and $f_{21}^{(3)}$

$$f_{11}^{(3)} = \frac{n_{(3)}}{n_G} \sum_g \left(D_{11}^{(3)}(g)\right)^* O_g f$$

$$f_{11}^{(3)} = \frac{2}{6}\Big(f - \frac{1}{2} O_{r_1} f - \frac{1}{2} O_{r_2} f + O_{s_0} f$$
$$- \tfrac{1}{2} O_{s_1} f - \tfrac{1}{2} O_{s_2} f\Big)$$

Where we read of the blue matrix elements from the table on the next page. Similarly

$$f_{21}^{(3)} = \frac{2}{6}\left(\frac{\sqrt{3}}{2} O_{r_1} f - \frac{\sqrt{3}}{2} O_{r_2} f + \frac{\sqrt{3}}{2} O_{s_1} f - \frac{\sqrt{3}}{2} O_{s_2} f\right)$$

- We also construct

$f_{12}^{(\mu)}$ and $f_{22}^{(\mu)}$

- The functions

$$f_{a1}^{(\mu)} = \{ f_{11}^{(\mu)} \quad f_{21}^{(\mu)} \}$$

Are partners in an irreducible rep $\mu = 3$. They span an <u>invariant</u> subspace in the span of the six original functions $\{ f, O_{r_1} f, O_{r_2} f, O_{s_0} f, O_{s_1} f, O_{s_2} f \}$

- Similarly $f_{a2}^{(\mu)} = \{ f_{12}^{(\mu)}, f_{22}^{(\mu)} \}$ are partners in an irrep.

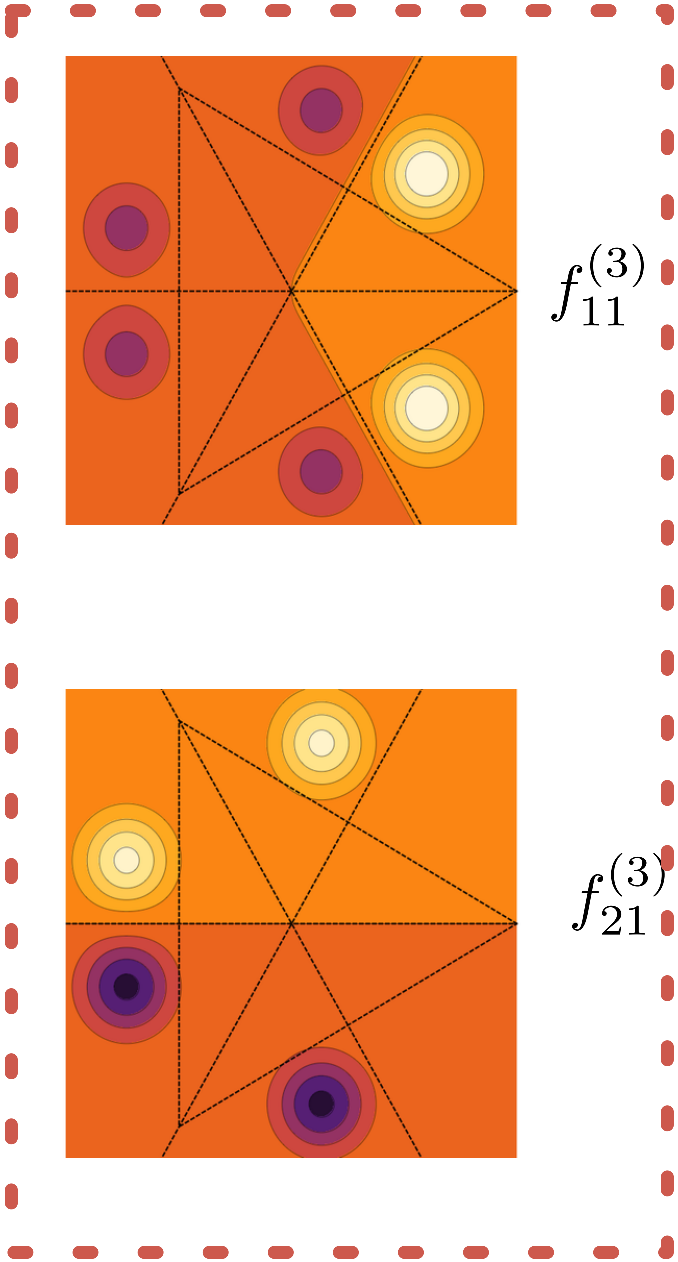
D_3 Matrix Representation Summary

	r_0	r_1	r_2	s_0	s_1	s_2
Identity	1	1	1	1	1	1
Alternate	1	1	1	-1	-1	-1
Matrix	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{\ }}{2} \\ -\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}$
Regular	$\mathbb{1}_{6\times 6}$	see multiplacation table				

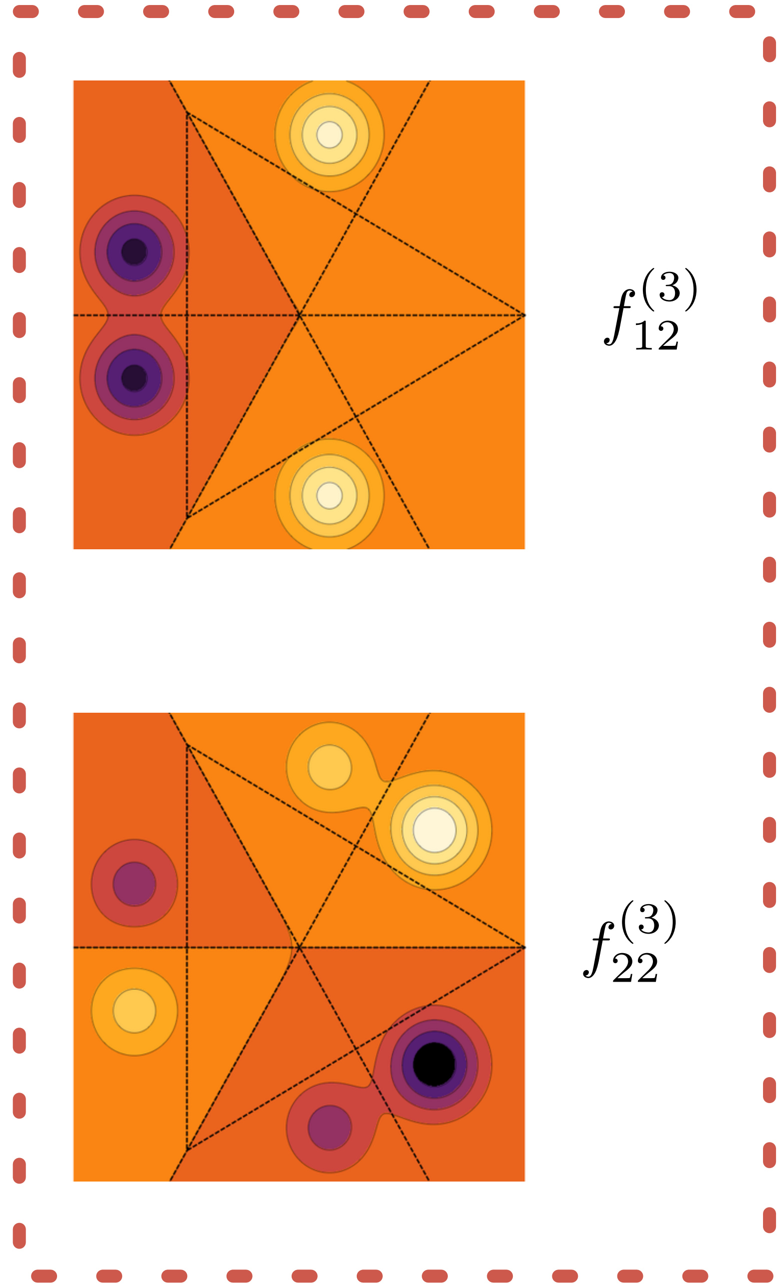
We used the circled entries to make $f^{(3)}_{a1}$

- Here $\mathbb{1}_{6\times 6}$ is the 6×6 identity matrix. Some matrix representatives of the regular representation are

$$D(r_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad D(s_0) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Partners in an irreducible rep



Partners in an irreducible rep

- The functions $f^{(\mu)}_{ab}$ are partners in a irreducable representation (μ). Take $b = 1$ for example. Being partners means

$$\boxed{O_g f^{(\mu)}_{ab} = f^{(\mu)}_{cb} \left(D^{(\mu)}_{ca}(g) \right)^*}$$

- What this means is that

$$f^{(3)}_{11}$$

can be transformed into $f^{(3)}_{21}$ by a linear combination of group operations. Indeed

$$f^{(3)}_{21} = \frac{1}{2} O_{r_1} f^{(3)}_{11} - \frac{1}{2} O_{r_2} f^{(3)}_{11}$$

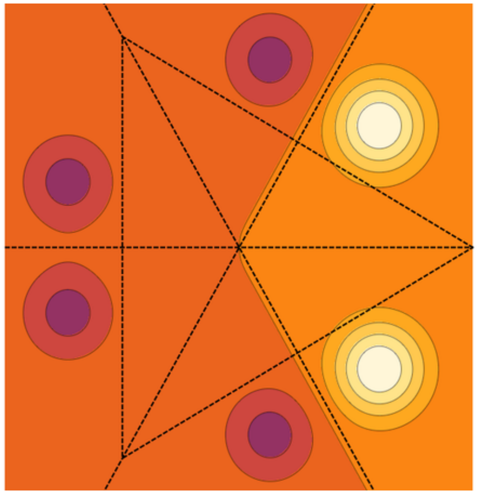
Similarly $f^{(\mu)}_{21}$ can be transformed to $f^{(3)}_{11}$.

- But no amount of group transformations will change $f^{(1)}_{11}$ to $f^{(3)}_{11}$. ($f^{(1)}_{11}$ is totally symmetric)

Partners in the same irrep are mixed by the group operations

$f_{11}^{(3)}$

(but different irreps do not mix)



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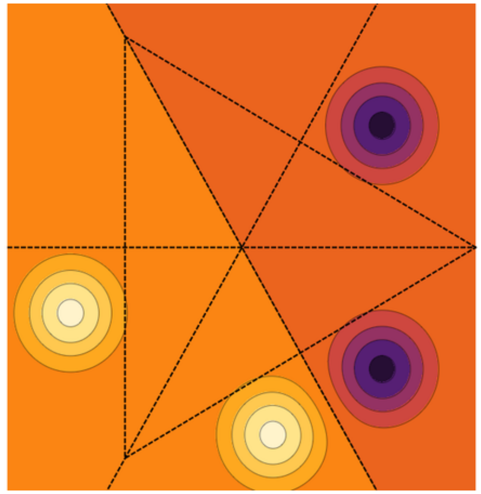
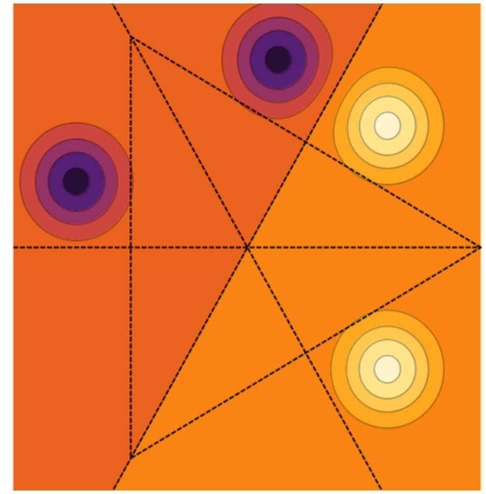
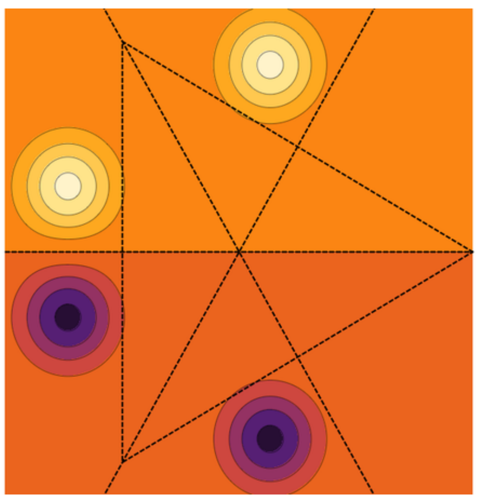


+ 1/2



$$f_{11}^{(3)} = \frac{1}{2}(-O_{r_1} + O_{r_2})f_{21}^{(3)}$$

$f_{21}^{(3)}$



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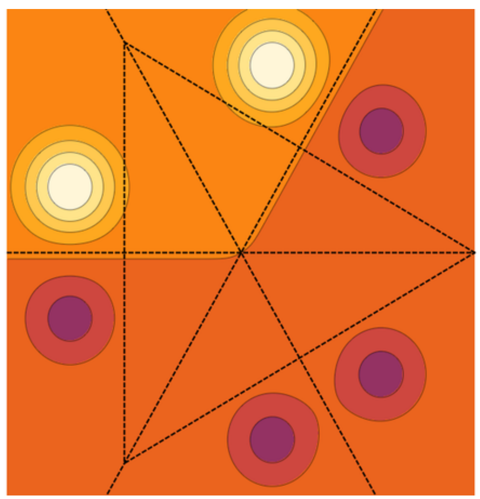

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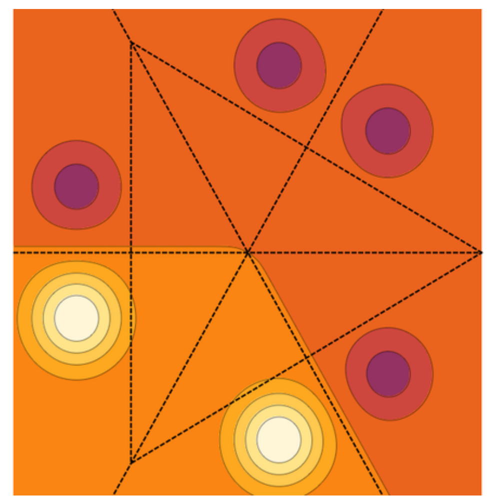


$$f_{21}^{(3)} = \frac{1}{2}(O_{r_1} - O_{r_2})f_{11}^{(3)}$$

• Proof that $f_{ab}^{(\mu)}$ are partners in an irrep:

$$f_{ab}^{(\mu)}(x) = \frac{n_\mu}{n_G} \sum_{g_1} D_{ba}^{(\mu)}(g_1^{-1}) \, O_{g_1} f$$

So

$$O_g f_{ab}^{(\mu)}(x) = \frac{n_\mu}{n_G} \sum_{g_1} D_{ba}^{(\mu)}(g_1^{-1}) \, O_g O_{g_1} f$$

$$= \frac{n_\mu}{n_G} \sum_{g_1} D_{ba}^{(\mu)}(g_1^{-1}) \, O_{gg_1} f$$

• Now as g_1 runs over the group so does $gg_1 \equiv g_2$

Define

$$g_2 = g g_1$$

$$g_2^{-1} = g_1^{-1} g^{-1}$$

$$g_2^{-1} g = g_1^{-1}$$

So

$$O_g f_{ab}^{(\mu)}(x) = \frac{n_\mu}{n_G} \sum_{g_2} D_{ba}^{(\mu)}(g_2^{-1} g) \, O_{g_2} f$$

$$= \frac{n_\mu}{n_G} \sum_{g_2} D_{bc}^{(\mu)}(g_2^{-1}) \, D_{ca}^{(\mu)}(g) \, O_{g_2} f(x)$$

$$= \left(\frac{n_\mu}{n_G} \sum_{g_2} D^{\mu}_{bc}(g_2^{-1}) \, O_{g_2} f \right) D_{ca}^{(\mu)}(g)$$

$$\boxed{O_g f^{\mu}_{ab} = f_{cb}^{(\mu)} \, D_{ca}^{(\mu)}(g)}$$

The Decomposition Into Elements of definite symmetry

- The operators $\hat{e}^{(\mu)}_{ab}$ form a complete basis for the group agebra $\hat{x} = \sum x_i \hat{g}_i$

- This means that any $\hat{g}_i$ can be expressed as a linear combo of $\hat{e}^{(\mu)}_{ab}$. In particular

$$\mathbb{1} = \sum_{\mu,a,b} c^{(\mu)}_{ab} \hat{e}^{(\mu)}_{ab} \qquad \hat{e}^{(\mu)}_{ab} = \frac{n_\mu}{n_G} \sum_g (D^{(\mu)}_{ab}(g))^* \hat{g}$$

Then since $\langle \hat{e}^{(\mu)}_{ab}, \hat{e}^{(\nu)}_{cd} \rangle = \frac{n_\mu}{n_G} \delta_{\mu\nu} \delta_{ac} \delta_{bd}$ we find

$$\langle \hat{e}^{(\mu)}_{ab}, \mathbb{1} \rangle = c^{(\mu)}_{ab} n_\mu / n_G$$

- Then for any "vector" $\hat{V} = \sum_i v_i \hat{g}_i$

$$\hat{V} = \sum_g V(g) \hat{g} = \sum_i V(g_i) \hat{g}_i = \sum V_i \hat{g}_i$$

We have

$$\langle \mathbb{1}, \hat{V} \rangle = V(\mathbb{1})$$ ← just the component of $\hat{V}$ along $\mathbb{1}$

$$\langle \hat{V}, \mathbb{1} \rangle = V^*(\mathbb{1})$$ ← the complex conjugate

Thus

component of $\hat{e}$ along $\mathbb{1}$

$$c^{(\mu)}_{ab} = \frac{n_G}{n_\mu} \langle \hat{e}^{(\mu)}_{ab}, \mathbb{1} \rangle = \frac{n_G}{n_\mu}\left(\frac{n_\mu}{n_G} D^{(\mu)}_{ab}(\mathbb{1})\right)$$

group op

$$C^{(\mu)}_{ab} = D^{(\mu)}_{ab}(\mathbb{1})$$

$$C^{(\mu)}_{ab} = \delta_{ab}$$

The matrix of $\mathbb{1}$ is the identity matrix

$$\mathbb{1}^{(\mu)}_{ab} = \delta_{ab}$$

So

$$\mathbb{1} = \sum_{\mu a, b} \delta_{ab} \hat{e}^{(\mu)}_{ab}$$

$$\boxed{\mathbb{1} = \sum_{\mu} \sum_{a} \hat{e}^{(\mu)}_{aa}}$$

Then we may decompose any function into components

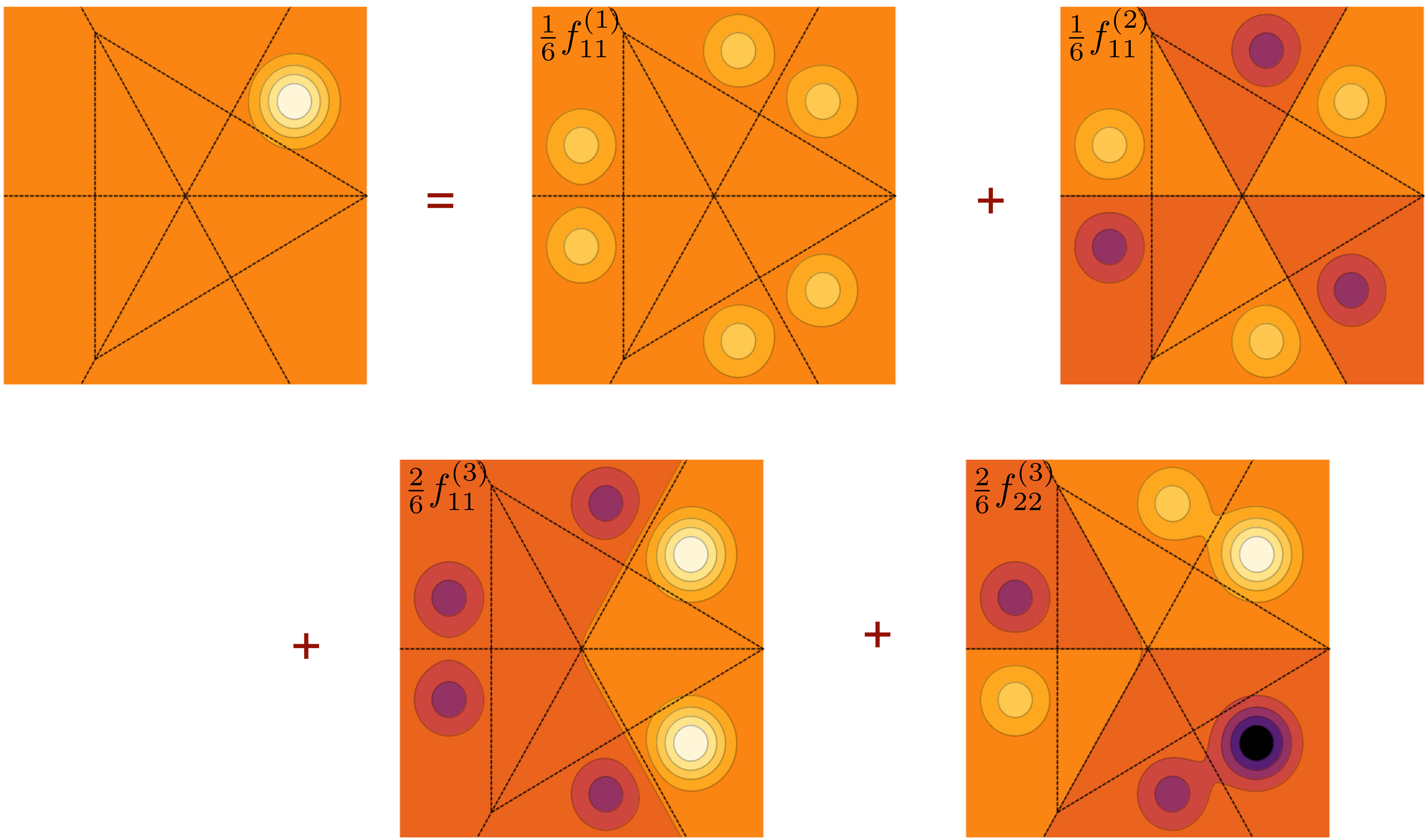
$$f = \mathbb{1} f$$

$$\boxed{f = \sum_{\mu} \sum_{a} \hat{e}^{(\mu)}_{aa} f = \sum_{\mu} \sum_{a} f^{(\mu)}_{aa}}$$

- The $f^{(\mu)}_{aa}$ transform as a row of an irreducible representation.

This is portrayed graphically on the slide.

Projection theorem portrayed graphically



$$f = \sum_{\mu,a} f_{aa}^{(\mu)} = f_{11}^{(1)} + f_{11}^{(2)} + f_{11}^{(3)} + f_{22}^{(3)}$$