Conjugacy Classes

- A group element y is conjugate to x if there is a $g \in G$ such that

$$y = g x g^{-1}$$

We write $y \sim x$

- Then this is an equivalence relation

  - If $y \sim x$ the $x \sim y$
  - If $z \sim y$ and $y \sim x$, then $z \sim x$

We can divide the group into conjugacy classes C, i.e the set of elements conjugate to each other. Note

$$C_1 = \{ \mathbb{1} \}$$

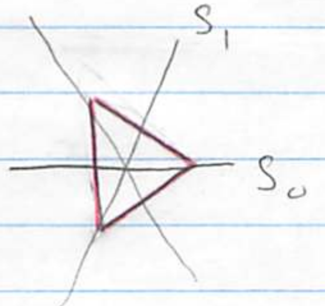
$$C_2 = \{ \hat{r}_1, \hat{r}_2 \} \leftarrow \text{rotations in a class}$$

$$C_3 = \{ \hat{s}_0, \hat{s}_1, \hat{s}_2 \} \leftarrow \text{reflection in a class}$$

e.g.

$$\hat{s}_1 = (\hat{r}_1)^{-1} s_0 \hat{r}_1$$

Think about it pictorially



- I can achieve an $\hat{S}_1$ reflection by making a rotation, reflecting over $\hat{S}_0$, and reflecting back
rotating

- The notation of conjugacy gives mathematical rigor to the intuitive notion that all reflections are similar somehow, and all rotations are similar somehow

- We will label the conjugacy classes as

$$C_I \quad \text{with} \quad I = 1 \ldots N_{class}$$

- See previous page for C_I of D_3. The number of elements in each class is n_I

$$n_{I=1} = 1$$

e.g.

$$n_{I=2} = 2 \leftarrow \text{two elements in } C_2 = \{\hat{r}_1, \hat{r}_2\}$$

$$n_{I=3} = 3$$

(the fact that n_{I}=I in this specific case, has no significance)

Character Analysis

- The character of a group rep $\chi(g) = \mathrm{Tr}\, D(g)$ is the same for each member of a conjugacy class, since if $y \sim z$

$$\chi(y) = \mathrm{Tr}\, D(y)$$

$$= \mathrm{Tr}\, D(g z g^{-1})$$

$$= \mathrm{Tr}\left[D(g)\, D(z)\, D^{-1}(g)\right]$$

$$= \mathrm{Tr}\left[D^{-1}(g)\, D(g)\, D(z)\right] = \mathrm{Tr}\, D(z)$$

- Thus we may list the characters of a group in a reduced table

$\leftarrow C_I(n_I)$ the n_I is the number of class elements

		$C_1(1)$ r_0	$C_2(2)$ $r_1 r_2$	$C_3(3)$ $s_0 s_1 s_2$
$\mu = 1$	Identity	1	1	1
$\mu = 2$	Alternating	1	1	-1
$\mu = 3$	Matrix	2	-1	0

For instance $\mathrm{Tr}\, D^{(3)}(r_1) = \mathrm{Tr} \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ +\sqrt{3}/2 & -1/2 \end{pmatrix} = -1$

see list of matrices

Character Orthogonality

- Given the orthogonality theorem

$$\sum_g D^{(\mu)}_{ab}(g) \left(D^{(\nu)}_{cd}(g) \right)^* = \frac{n_G}{n_\mu} \delta_{\mu\nu} \delta_{ac} \delta_{bd}$$

We may contract
ab and cd

$$\sum_g \sum_a \sum_c D^{(\mu)}_{aa}(g) \left(D^{(\nu)}_{cc}(g) \right)^* = \frac{n_G}{n_\mu} \delta_{\mu\nu} \overbrace{\sum_{a,c} \delta_{ac} \delta_{ac}}^{= n_\mu}$$

- Thus the characters of different reps are orthogonal

$$\sum_g \chi^{(\mu)}(g) \left(\chi^{(\nu)}(g) \right)^* = n_G \delta_{\mu\nu}$$

- But the character is only a function of the class (it is a so-called class function)

$$\sum_{I=1}^{N_{class}} \chi^{(\mu)}(C_I) \left(\chi^{(\nu)}(C_I) \right)^* n_I = n_G \delta_{\mu\nu}$$

This is a kind of inner product in class space, It says the rows are orthogonal

	C_1	C_2	C_3
$\mu=1$	1	1	1
$\mu=2$	1	1	-1
$\mu=3$	2	-1	0

e.g. the first and second row the orthogonality is

$$1 \cdot 1 + 1 \cdot 1 \cdot 2 + 1 \cdot (-1) \cdot 3 = 0$$

- The $\chi(C_I)$ are "vectors" $\vec{\chi}$ in class space $\{C_1, C_2, C_3\}$ which has dimension $N_{class} = 3$. For every rep $\mu = 1 \ldots N_{rep}$ we get another orthogonal vector $\vec{\chi}^{(\mu)}$

We must have

(or the number of orthogonal vectors in the "class space" would exceed the dimension of the space)

$$N_{rep} \leq N_{class}$$

- In fact it turns out that $N_{rep} = N_{class}$, In words, the number of irreducible representations is equal to the number of conjugacy classes.

- In fact the columns of the character table are orthogonal as well (summing over reps)

$$\sum_{\mu=1}^{N_{rep}} \chi^{(\mu)}(C_I)\left(\chi^{(\mu)}(C_J)\right)^* = \frac{n_G}{n_I}\delta_{IJ}$$

Thus contracting column 1 and 2

$$1 \cdot 1 + 1 \cdot 1 + 2 \cdot (-1) = 0$$

(See Hammermesh for proof)

Reduction of Representations

- Given a representation how can we tell if it is reducable?

$\swarrow$ the a_μ are integers

$$D(g) = \sum_\mu \oplus\, a_\mu D^{(\mu)} \qquad \text{e.g.}$$

$$= 2D^{(1)} \oplus 3D^{(2)} = \begin{pmatrix} D^{(1)} & & & & \\ & D^{(1)} & & & \\ & & D^{(3)} & & \\ & & & D^{(3)} & \\ & & & & D^{(3)} \end{pmatrix}$$

Well $\chi(g) = \mathrm{Tr}\, D(g)$

$$\chi(g) = \sum_\mu a_\mu \chi^{(\mu)}(g)$$

orthogonal " $\vec{\chi}^{(\mu)} \cdot \vec{\chi}^{(\nu)} \propto \delta_{\mu\nu}$ "

Since for irreps, $\sum_g \chi^{(\mu)}(g) (\chi^{(\nu)}(g))^* = n_G \delta_{\mu\nu}$

So

$$\sum_g \chi(g) \chi^*(g) = \left(\sum_\mu |a_\mu|^2 \right) n_G > n_G$$

- So we have a remarkably simple result: if a representation is irreducable, the sum of charecters squared is the order of the group, else it is reducable.

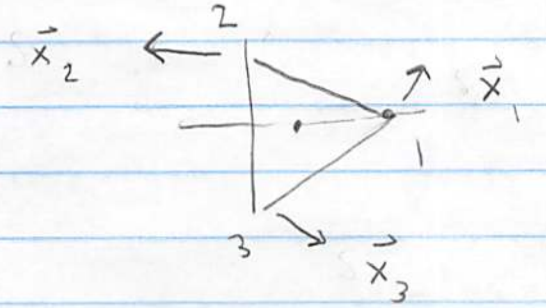
Further we may

Find the a_μ by taking the dot product of $\vec{\chi}(g)$ with $\chi^\mu(g)$

$$a_\mu = \frac{1}{n_G} \sum_g \chi(g) (\chi^\mu(g))^*$$

We will use this shortly

Small Oscillations



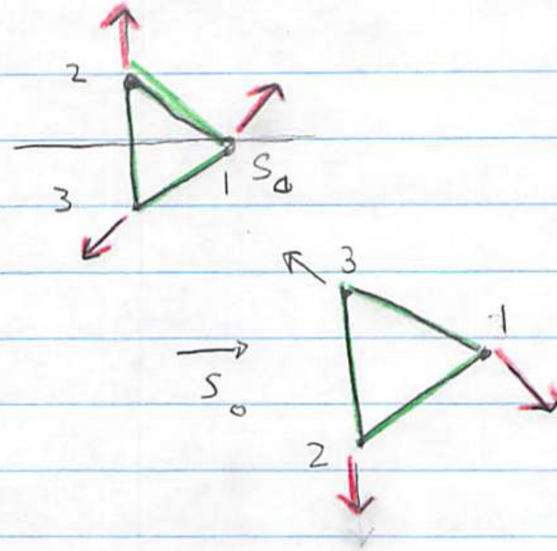
$\vec{x}_1, \vec{x}_2, \vec{x}_3$ are small

The vector space of displacements is

$$\vec{q} = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix} = q_a \vec{e}_a = (\vec{x}_1, \vec{x}_2, \vec{x}_3) \qquad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then clearly if I have a configuration, $\vec{q}$ then a rotated configuration will have the same kinetic and potential energies for any displacement. The lagrangian $\mathcal{L}[\vec{q}]$ is unchanged by group ops $\mathcal{L}[O_{r_0} \vec{q}]$ for any $\vec{q}$

Lets Look at $\hat{O}_{s_0} q$



$$\hat{O}_s q = \begin{pmatrix} x_1 \\ -y_1 \\ x_3 \\ -y_3 \\ x_2 \\ -y_2 \end{pmatrix}$$

Then $\quad q_a \xrightarrow[s_0]{} O_{ab} q^b$

$$O_{ab}^{(s_0)} = \left(\begin{array}{c|c|c} \begin{matrix}1 & \\ & -1\end{matrix} & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}1 & \\ & -1\end{matrix} \\ \hline 0 & \begin{matrix}1 & \\ & -1\end{matrix} & 0 \end{array}\right) = \left(\begin{array}{c|c|c} D^{(3)} & 0 & 0 \\ \hline 0 & 0 & D^{(3)} \\ \hline 0 & D^{(3)} & 0 \end{array}\right)$$

$$D^{(3)}(s_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Where

$D^{(3)}$ is the 2×2 matrix rep that describes how D_3 acts on vectors. Thus the group operations involve the action of $D^{(3)}$ on the vectors together with a permutation of positions

- We may write down the other ops in a similar fashion, e.g.

$$\mathcal{O}(r_1) = \begin{pmatrix} 0 & 0 & D \\ D & 0 & 0 \\ 0 & D & 0 \end{pmatrix} \quad \text{where } D = D^{(3)}(r_1) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$\mathcal{O}(e) = \begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \quad \text{where } D = D^{(3)}(\mathbb{1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- The complete list is in the slides. Already using $\mathcal{O}(e)$, $\mathcal{O}(r_1)$, $\mathcal{O}(s_0)$ we can determine the characters

$$\underbrace{\chi_0(e) = 6}_{C_1} \qquad \underbrace{\chi(r_1) = 0}_{C_2} \qquad \underbrace{\chi(s_0) = 0}_{C_3}$$

$$n_1 = 1 \qquad n_2 = 1 \qquad n_1 = 3$$

- Then this rep is reducable

$$\sum_g |\chi(g)|^2 = 6 \cdot 6 > 36$$

$$O_{\mathbb{1}} = \begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \quad O_{r_1} = \begin{pmatrix} 0 & 0 & D \\ D & 0 & 0 \\ 0 & D & 0 \end{pmatrix} \quad O_{r_2} = \begin{pmatrix} 0 & D & 0 \\ 0 & 0 & D \\ D & 0 & 0 \end{pmatrix}$$

$$O_{s_0} = \begin{pmatrix} 0 & 0 & D \\ D & 0 & 0 \\ 0 & D & 0 \end{pmatrix} \quad O_{s_1} = \begin{pmatrix} D & 0 & 0 \\ 0 & 0 & D \\ 0 & D & 0 \end{pmatrix} \quad O_{s_2} = \begin{pmatrix} 0 & 0 & D \\ 0 & D & 0 \\ D & 0 & 0 \end{pmatrix}$$

where D in O_{s_2} for example is short for $D^{(3)}(s_2)$.

	r_0	r_1	r_2	s_0	s_1	s_2
$D^{(3)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}$

By a change of basis we may decompose

$$D(g) \to \underline{D}(g) = S^{-1} D(g) S$$

As a direct sum of irreps

$$\underline{D}(g) = \sum D^{(\mu)} a_\mu$$

Then

$$a_\mu = \frac{1}{n_G} \sum_I \chi(C_I) \left(\chi^{(\mu)}(C_I)\right)^* n_I$$

$$a_{(1)} = \frac{1}{6} \, 6 \cdot 1 = 1$$

$$a_{(2)} = \frac{1}{6} \, 6 \cdot 1 = 1$$

$$a_{(3)} = \frac{1}{6} \, 6 \cdot 2 = 2$$

So after a change of basis we must have

$$\underline{D}(g) = D^{(1)} \oplus D^{(2)} \oplus 2 D^{(3)}$$

$$\underline{D} = \left(\begin{array}{c|c|c|c} 1\times1 & & & \\ \hline & 1\times1 & & \\ \hline & & 2\times2 & \\ \hline & & & 2\times2 \end{array} \right) = 6\times6 \text{ matrix}$$

it will be easier to analyze the small

Oscillations in this basis

The mechanical problem

- $T = \frac{1}{2} \sum_a m \dot{q}_a^2$

- $U = U_0 + \frac{1}{2} \frac{\partial^2 U}{\partial q_a \partial q_b} q_a q_b$

$$= \frac{1}{2} q_a H_{ab} q_b + \text{const} \qquad H_{ab} \equiv \frac{\partial^2 U}{\partial q^a \partial q^b}$$

Some notation $\vec{q} \equiv \left(\right)$

- $\langle \vec{q}, \vec{q} \rangle \equiv \vec{q}^T \vec{q}$

Then

$$\langle \vec{q}, H\vec{q} \rangle \equiv \vec{q}^T H \vec{q}$$

- The mechanical problem is

$$m \frac{d^2 \vec{q}}{dt^2} + H \vec{q} = 0$$

- Our goal is to find the eigenvectors and evalues of H

$$H \vec{\psi}_\lambda = k_\lambda \vec{\psi}_\lambda$$

Then we may solve the EOM. For if

$$\vec{q} = q_\lambda \vec{\psi}_\lambda ,$$

Then

$$m \frac{d^2 q_\lambda}{dt^2} = - k_\lambda q_\lambda \qquad q_\lambda(t) = q_\lambda(0)\, e^{\pm i \omega_\lambda t}$$

with $\omega_\lambda = \sqrt{k_\lambda / m}$.

The eigenvectors of H are real and orthogonal.

Symmetry:

- Under the group ops $\vec{q} = q_a \vec{e}_a$

$$q_a \xrightarrow{g} O_{ba}(g)\, q_a \qquad \vec{q} \rightarrow O_g \vec{q}$$

- The matrices O are orthogonal $O^T O = \mathbb{1}$.
  So $\langle O\vec{x}, O\vec{y} \rangle = \langle \vec{x}, O^T O \vec{y} \rangle$
- The potential energy is unchanged by the group operations

$$\langle O\vec{q}, H O\vec{q} \rangle = \vec{q}^{\,T} O^T H O \vec{q} = \langle \vec{q}, H\vec{q} \rangle$$

So we must have

$$O^T H O = H \qquad \text{i.e} \qquad O^{-1} H O = H$$

or $[H, O] = 0$

How does this help us?

- Suppose that $\vec{\psi}$ is an eigen-function. Then $O_g \vec{\psi}$ is also an eigen-function with the same eigenvalue

$$O_g \, H \vec{\psi} = k \, O_g \vec{\psi}$$

commute

$$H \, O_g \psi = k \, O_g \psi$$

- So we get a set of functions

$$\{ \vec{\psi}, O_{r_1} \vec{\psi}, O_{r_2} \vec{\psi}, O_{s_0} \vec{\psi}, O_{s_1} \vec{\psi}, O_{s_2} \vec{\psi} \}$$

These may not be distrinct. But further group action will simply mix up these states

- There are a set of λ functions which have common eigenvalue k_λ

$$\{ \vec{\psi}_1 \ldots \vec{\psi}_\lambda \}$$

(they span a space which is invariant under group operations)

Which span this space. • Action by the group Returns a linear combo of these states

$$O_g \vec{\psi}_\lambda = \vec{\psi}_{\lambda'} D_{\lambda \lambda'}(g)$$

The $D_{\lambda\lambda'}(g)$ is a representation of the group.

In the "normal" situation

$D_{\lambda\lambda'}$

Will be an irreducable representation $^{(\mu)}$ of the group. The $\psi_\lambda^{(\mu)}$ are degenerate because of symmetry, we can change $\psi_1^{(\mu)}$ to (linear combos of) $\psi_2^{(\mu)}$ by group ops.
(and the group ops commute with hamiltonian)

- If the representation is reducable then we could decompose it into irreps and have a set $\{\psi_a^{(\mu_1)}, \psi_a^{(\mu_2)}\}$ all of which have the same eigenvalue. The degeneracy between (μ_1) functions and (μ_2) functions would seem "accidental" as there is no group op which connects the (μ_1) fcns with the (μ_2) fcns. Typically this indicates that there is a larger symmetry group which connects (μ_1) to (μ_2).
(though accidents can actually happen, and sometimes in nature two levels are close in value for no good reason)