Problem 1. Levi-civita practice

(a) Using the Levi-civitia tensor, show that for a constant field magnetic $\boldsymbol{B}$ field show that the vector potential ($\boldsymbol{B} = \nabla \times \boldsymbol{A}$) can be written:

$$\boldsymbol{A} = \tfrac{-1}{2}\boldsymbol{r} \times \boldsymbol{B} \tag{1}$$

(b) Show (using the Levi-civita symbol) and $\epsilon_{ijk}\epsilon^{ijk} = 3!$ that

$$\det(\mathbb{A}) = \det(\mathbb{A}^T) \tag{2}$$

and

$$\det(\mathbb{A}\mathbb{B}) = \det(\mathbb{A})\det(\mathbb{B}) \tag{3}$$

(c) When differentiating $1/r$ we write

$$\frac{1}{r} = \frac{1}{\sqrt{x^i x_i}} \tag{4}$$

with $\boldsymbol{x} = x^i \boldsymbol{e}_i$, and use results like

$$\partial_i x^j = \delta_i^j \qquad \partial_i x_j = \delta_{ij} \qquad \partial_i x^i = \delta_i^i = d = 3 \tag{5}$$

where $d = 3$ is the number of spatial dimensions. (It is usually helps to write this as d rather than 3 to get the algebra right). In this way, one finds that field due to an electric point charge (monopole) is the familiar $\hat{\boldsymbol{r}}/r^2$. Go through these steps!

$$j\text{-th component of} \quad -\nabla(1/r) = \left(-\nabla\frac{1}{r}\right)_j = -\partial_j \frac{1}{\sqrt{x^i x_i}} = \frac{\frac{1}{2}(x^i\delta_{ji} + x_i\delta^i_j)}{(x^k x_k)^{3/2}} = \frac{x_j}{r^3} = \frac{(\boldsymbol{n})_j}{r^2} \tag{6}$$

where $\hat{\boldsymbol{r}} \equiv \boldsymbol{n} = \boldsymbol{r}/r$. In general

$$\partial_j \, r^\alpha = \alpha \, r^{\alpha-1} \, n_j \, . \tag{7}$$

Using tensor notation (*i.e.* indexed notation) show that

$$\nabla \times \frac{\hat{\boldsymbol{r}}}{r^2} = 0 \tag{8}$$

This verifies that the $\nabla \times \boldsymbol{E} = 0$ for a point charge.

(d) The vector potential of a magnetic dipole is

$$\boldsymbol{A} = \frac{\boldsymbol{m} \times \boldsymbol{n}}{4\pi r^2} \tag{9}$$

where $\boldsymbol{m}$ is a constant vector known as the magnetic dipole moment and $\boldsymbol{n} = \boldsymbol{r}/r$. Recall that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Using the tensor notation (*i.e.* indexed notation) show that

$$\boldsymbol{B} = \frac{3(\boldsymbol{n} \cdot \boldsymbol{m})\boldsymbol{n} - \boldsymbol{m}}{4\pi r^3} \tag{10}$$

Problem 2. Easy important application of Helmholtz theorems

(a) We showed in class that the source free Maxwell equations (*i.e.* those without ρ and $\boldsymbol{j}$), are solved by writing $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of a scalar field Φ (the scalar potential) and a vector field $\boldsymbol{A}$ (the vector potential)

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{11}$$

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\Phi \tag{12}$$

Now, using the sourced Maxwell equations (*i.e.* those with ρ and $\boldsymbol{j}$), show that current must obey the conservation Law

$$\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0\,, \tag{13}$$

to be consistent with the Maxwell equations.

Problem 3. The multipole expansion and a rotating quadrupole

Consider a charge density $\rho(\boldsymbol{x})$ as shown below. The potential is given by

$$\phi(\boldsymbol{r}) = \int d^3\boldsymbol{x} \frac{\rho(\boldsymbol{x})}{4\pi|\boldsymbol{r} - \boldsymbol{x}|} \tag{14}$$

The multipole expansion determines the potential far from the charges, i.e. for $\boldsymbol{r} \gg \boldsymbol{x}$.

(a) Show that for $\boldsymbol{r} \gg \boldsymbol{x}$ we have to quadratic order in $\boldsymbol{x}$ (or more formally x^i/r) the expansion

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{x}|} = \frac{1}{\sqrt{r^2 - 2\boldsymbol{r}\cdot\boldsymbol{x} + \boldsymbol{x}\cdot\boldsymbol{x}}} \simeq \frac{1}{r} + \frac{r_i x^i}{r^3} + \frac{r_i r_j}{2r^4}(3x^i x^j - x^2\,\delta^{ij}) + \ldots \tag{15}$$

where $x^2 = \boldsymbol{x}\cdot\boldsymbol{x} = x_\ell x^\ell$, and we are using some power series expansions which all physics students must know by heart but don't always[1].

Using this expansion, confirm for yourself (but do not bother turning in!) that the potential far from a charge distribution takes the form

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi}\left[\frac{q_{\rm tot}}{r} + \frac{p^i n_i}{r^2} + \frac{1}{2}\frac{\mathcal{Q}^{ij} n_i n_j}{r^3} + \ldots\right] \tag{18}$$

Here $n_i = r_i/r$ is unit vector in the direction of $\boldsymbol{r}$, and the momopole, dipole, and quadrupole moments, are respectively

$$q_{\rm tot} = \int_V d^3\boldsymbol{x}\,\rho(\boldsymbol{x}) \tag{19}$$

$$p^i = \int_V d^3\boldsymbol{x}\,\rho(\boldsymbol{x}) x^i \tag{20}$$

$$\mathcal{Q}^{ij} = \int_V d^3\boldsymbol{x}\,\rho(\boldsymbol{x})\,(3x^i x^j - x^2\delta^{ij}) \tag{21}$$

(b) Consider four point charges of charge $+q, -q, +q, -q$ arranged in a square of side $2a$ lying flat in the x, y plane. The square is rotating with constant angular velocity ω in a counter clockwise fashion, and at time $t = 0$ is in the configuration shown below.

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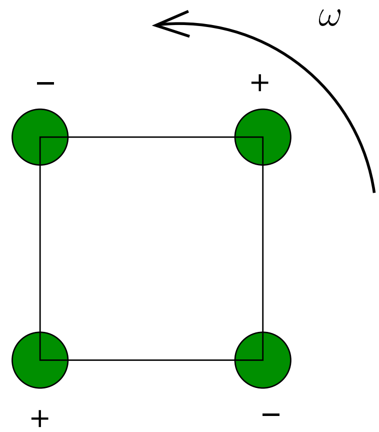
[1]

$$(1+z)^\alpha \simeq 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \ldots \tag{16}$$

$$\log(1+z) \simeq z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots \tag{17}$$



(i) Determine all components of the quadrupole tensor at $t = 0$.

(ii) In class we said that under rotations the components of the quadrupole tensor transform as

$$\underline{\mathcal{Q}}^{ij} = R^{i}_{\ \ell} R^{j}_{\ m} \mathcal{Q}^{\ell m} \tag{22}$$

Use this transformation rule to show that the components of the quadrupole tensor as a function of time are given by

$$(\mathcal{Q})^{ij} = 12qa^2 \begin{pmatrix} -\sin(2\omega t) & \cos(2\omega t) & 0 \\ \cos(2\omega t) & \sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{23}$$

Give a one (or at most two) sentence explanation why $\mathcal{Q}^{xx}$ is negative when $\omega t = \frac{\pi}{4}$ Next semester you may be asked to use this result to determine the power radiated by such a rotating array of charges.

Problem 4. Integration by Parts (IBP) like mad!

Answer briefly

(a) Assume ϕ and $|\boldsymbol{G}|$ fall faster than $1/r$ as $\boldsymbol{r} \to \infty$. Let $\boldsymbol{F} = \nabla\phi$ and $\nabla \times \boldsymbol{G} = 0$, use indices and IBP like mad to show that $\int \mathrm{d}^3x \, \boldsymbol{F} \times \boldsymbol{G} = 0$

(b) Consider a current density $\boldsymbol{j}$ entirely contained within a volume V. The current is steady and therefore satsifies $\nabla \cdot \boldsymbol{j} = 0$. Show that[2]

$$\int_V d^3\boldsymbol{x} \, j^\ell(\boldsymbol{x}) = 0 \tag{24}$$

(c) Consider a two dimensional surface S bounded by a loop C. Show that[3]

$$\int dS_i = \frac{1}{2} \oint_C (\boldsymbol{r} \times \mathrm{d}\boldsymbol{\ell})_i \tag{25}$$

Here the magnitude $\mathrm{d}\boldsymbol{S}$ is the area of the infinitessimal surface element. $\mathrm{d}\boldsymbol{S}$ points normal to the surface.

(d) Let S be the surface that bounds a volume V. Show that

$$\oint dS_i = 0 \tag{26}$$

and

$$\frac{1}{3} \oint \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{r} = V \tag{27}$$

[2]Hint: write $j^i \delta_i^\ell = j^i \partial_i x^\ell$ and IBP like mad!

[3]Hint: use the results probelm 1(a) for the constant vector $\boldsymbol{e}_i$.