Problem 1. Volumes and dual bases

(a) Show that if

$$\epsilon_{abc} = \sqrt{g}[abc] \tag{1}$$

Then show ϵ^{abc} (defined from ϵ_{abc} by raising indices, e.g. $v^a = g^{ab}v_b$) is

$$\frac{1}{\sqrt{g}}[abc] \tag{2}$$

(b) Consider three vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$, which span a parallel piped of volume $\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) > 0$. The Gram-Schmidt decompostion constructs a set of orthogonal vectors from $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$

$$\boldsymbol{b}_1 = \boldsymbol{a}_1 \tag{3}$$

$$\boldsymbol{b}_2 = \boldsymbol{a}_2 - \frac{(\boldsymbol{a}_2 \cdot \boldsymbol{b}_1)}{\boldsymbol{b}_1 \cdot \boldsymbol{b}_1}\boldsymbol{b}_1 \tag{4}$$

$$\boldsymbol{b}_3 = \boldsymbol{a}_3 - \frac{(\boldsymbol{a}_3 \cdot \boldsymbol{b}_1)}{\boldsymbol{b}_1 \cdot \boldsymbol{b}_1}\boldsymbol{b}_1 - \frac{\boldsymbol{a}_3 \cdot \boldsymbol{b}_2}{\boldsymbol{b}_2 \cdot \boldsymbol{b}_2}\boldsymbol{b}_2 \tag{5}$$

(i) Briefly interpret the decompisition graphically, and show that $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ are orthogonal.

(ii) Show using the properties of determinants that $\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \det(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)$ and that $\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = |\boldsymbol{b}_1||\boldsymbol{b}_2||\boldsymbol{b}_3|$. No long proofs please – just a fiew lines. Briefly interpret graphically.

(c) Consider given three basis vectors $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3$.

(i) Show that the dual basis is

$$\boldsymbol{g}^1 = \frac{\boldsymbol{g}_2 \times \boldsymbol{g}_3}{\omega} \qquad \boldsymbol{g}^2 = \frac{\boldsymbol{g}_3 \times \boldsymbol{g}_1}{\omega} \qquad \boldsymbol{g}^3 = \frac{\boldsymbol{g}_1 \times \boldsymbol{g}_2}{\omega} \tag{6}$$

where $\omega = \boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3)$.

(ii) Using the properties of the dual basis and determinants, show (in no more than three lines!) that $\Omega = \boldsymbol{g}^1 \cdot (\boldsymbol{g}^2 \times \boldsymbol{g}^3) = 1/\omega$.

Problem 2. A dot product in non-orthogonal coordinates

Consider a 2d-coordinate system

$$x = u^1 + 2u^2 \tag{7}$$
$$y = u^2 + u^1 \tag{8}$$

(a) Given the components of two vectors $v_a = (v_1, v_2)$ and $w_a = (w_1, w_2)$ so that $\boldsymbol{v} = v_a \boldsymbol{g}^a$ etc, explictly determine the dot product $\boldsymbol{v} \cdot \boldsymbol{w}$ in terms of these (lower) components.

Problem 3. Spherical coordinates

Spherical coordinates are defined by

$$x = r \sin\theta \cos\phi \tag{9}$$
$$y = r \sin\theta \sin\phi \tag{10}$$
$$z = r \cos\theta \tag{11}$$

(a) Determine the basis vectors $\boldsymbol{g}_r, \boldsymbol{g}_\theta, \boldsymbol{g}_\phi$ as an expansion in cartesian basis vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$, and illustrate them graphically.

(b) Determine the metric tensor g_{ab} and ds^2, and $\boldsymbol{g}^r, \boldsymbol{g}^\theta, \boldsymbol{g}^\phi$.

(c) Determine the volume measure dV using g_{ab}.

(d) Compute all Christoffel symbols by computing derivatives, e.g. compute

$$\partial_\theta \boldsymbol{g}_r \tag{12}$$

and reexpand the result in $\boldsymbol{g}_r, \boldsymbol{g}_\phi, \boldsymbol{g}_\theta$. Give a graphical explanation for the ratio of $\Gamma^\theta_{\phi\phi}$ to $\Gamma^r_{\phi\phi}$.

(e) Compute $\Gamma^r_{\phi\phi}$ and $\Gamma^\theta_{\phi\phi}$ using the famous formula

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab} \right) . \tag{13}$$

and verify that this agrees with the results in the previous item

(f) Consider cylindrical coordinates (look at lecture notes). Every year on the comps, some tragicomical[1] student writes

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \cos\phi \, \hat{\boldsymbol{\rho}} = 0 \qquad \text{crazily wrong!} \tag{14}$$

Show that the correct result is $\frac{1}{2}\hat{\mathbf{x}}$.

[1]Definition of tragicomic. 1 : of, relating to, or resembling tragicomedy. 2 : manifesting both tragic and comic aspects.

(g) The curl of vector field is

$$\nabla \times \boldsymbol{A} = \boldsymbol{e}_i \epsilon^{ijk} \partial_j A_k \tag{15}$$

Given this definition in cartesian coordinates, show by coordinate transformation that in a general coordinate system

$$\nabla \times \boldsymbol{A} = \boldsymbol{g}_a \epsilon^{abc} \nabla_b A_c \tag{16}$$

Argue that for the curl (and only the curl!) we may use a partial instead of covariant derivative

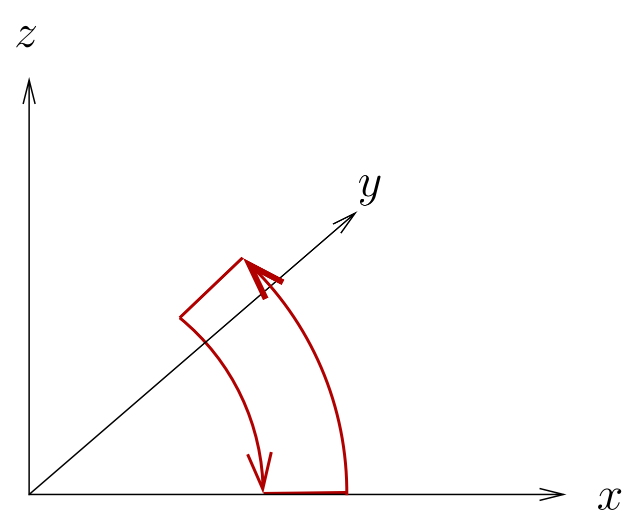
$$\nabla \times \boldsymbol{A} = \boldsymbol{g}_a \epsilon^{abc} \partial_b A_c \tag{17}$$

and use this result to show that for general orthogonal coordiantes

$$(\nabla \times \boldsymbol{V}) = \frac{\boldsymbol{e}_{\hat{1}}}{h_2 h_3}\left[\frac{\partial(h_3 V^{\hat{3}})}{\partial u^2} - \frac{\partial(h_2 V^{\hat{2}})}{\partial u^3}\right] + \frac{\boldsymbol{e}_{\hat{2}}}{h_1 h_3}\left[\frac{\partial(h_1 V^{\hat{1}})}{\partial u^3} - \frac{\partial(h_3 V^{\hat{3}})}{\partial u^2}\right] + \frac{\boldsymbol{e}_{\hat{3}}}{h_1 h_2}\left[\frac{\partial(h_2 V^{\hat{2}})}{\partial u^1} - \frac{\partial(h_1 V^{\hat{1}})}{\partial u^2}\right] \tag{18}$$

and here $\boldsymbol{e}_{\hat{a}} = \boldsymbol{g}_a / h_a$.

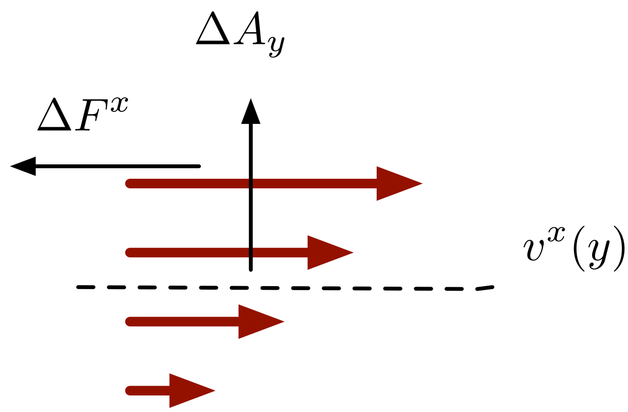
(h) For the specific surface shown below (i.e. the surface bounded by the red contour at $\phi = 0$ which forms a square in r, θ space), use Eq. (18) to prove the Stokes theorem for this specific surface



(i) • Recall that for a viscous fluid the force per area $\Delta F / \Delta A$ of two streams flowing past each other with different velocities Δv^x is

$$\frac{\Delta F^x}{\Delta A_y} = -\eta \frac{\Delta v^x}{\Delta y} \tag{19}$$

Here ΔF^x is the force on the upper (faster) stream by the lower (and slower) stream



I put a y-index on ΔA_y to indicate that area vector we are considering, $\Delta\vec{A} = \Delta A\,\vec{n}$, is pointing in the y direction.

- The force per area *defines* the stress tensor in the system. Thus, the stress tensor for the viscous fluid we have described above has the nonvanishing component

$$T^{xy} = -\eta\frac{\Delta v^x}{\Delta y} \tag{20}$$

Indeed, the stress tensor T^{ij} is very generally interpretted as

$$T^{ij} = \frac{\text{force in the } i\text{-th direction}}{\text{area in the } j\text{-th direction}} = \frac{\Delta F^i}{\Delta A_j} \tag{21}$$

The stress tensor in cartesian coordinates of a viscous fluid is

$$T^{ij}_{\rm visc} = -\eta(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_\ell v^\ell) \tag{22}$$

In the simple case where the x-velocity is a function of y (and all other velocity components vanish), $T^{xy} = -\eta\partial^y v^x$ is the only non-vanishing component.

- According to Landau and Lifshitz Fluid Mechanics (a standard text *not* on general relativity, which therefore uses normalized coordinate vectors), the divergence of the the velocity is (but they leave off the hats!)

$$\nabla\cdot\boldsymbol{v} = \frac{1}{r^2}\partial_r(r^2 v^{\hat r}) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta v^{\hat\theta}) + \frac{1}{r\sin\theta}\partial_\phi v^{\hat\phi} \tag{23}$$

Derive this results using the formula involving covariant derivatives, $\nabla_a v^a$. Also compute it using the general expression

$$\nabla\cdot\boldsymbol{v} = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}v^a) \tag{24}$$

According to Landau and Lifshitz one of the stress tensor components of a viscous fluid are $T^{\hat r\hat\theta}$ is (but they leave off the hats!)

$$T^{\hat r\hat\theta} = -\,\eta\left(\partial_r v^{\hat\theta} + \frac{1}{r}\partial_\theta v^{\hat r} - \frac{v^{\hat\theta}}{r}\right) \tag{25}$$

Derive this result, given its cartesian counter part, Eq. (22)

(j) Using the setup of the previous problem, suppose that at an angle θ, but $\phi = 0$, $T^{\hat{r}\hat{\theta}} = T^{\hat{\theta}\hat{r}}$ is the only non-vanishing component. What are the only non-vanishing cartesian components T^{ij} and how are they related to $T^{\hat{r}\hat{\theta}}$? Draw a picture to explain your result.