Spherical Coordinates and the Shear Stress

a) $x^1 = r\sin\theta\cos\phi$

$x^2 = r\sin\theta\sin\phi \qquad \vec{s} = x^1\hat{x} + x^2\hat{y} + x^3\hat{z}$

$x^3 = r\cos\theta$

Then

$\hat{x}$ component $\qquad$ $\hat{z}$ component

$$\vec{g}_r = \frac{\partial\vec{s}}{\partial r} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

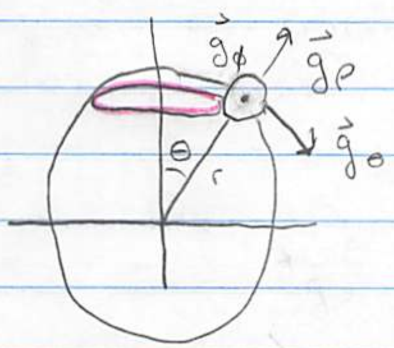
$\hat{y}$ component

$$\vec{g}_\theta = \frac{\partial\vec{s}}{\partial\theta} = (r\cos\theta\cos\phi,\ r\cos\theta\sin\phi,\ -r\sin\theta)$$

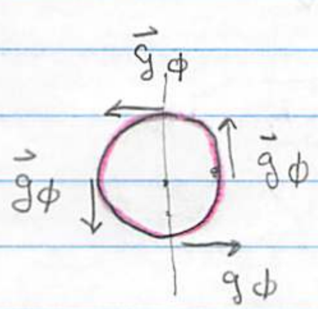
$$\vec{g}_\phi = (-r\sin\theta\sin\phi,\ r\sin\theta\cos\phi,\ 0)$$

Take $\phi = 0$

side view



top view



The circle has radius $r\sin\theta$

Then

b) $g_{ab} = \vec{g}_a \cdot \vec{g}_b$

- $g_{rr} = 1 = (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta$
- $g_{\theta\theta} = r^2$
- $g_{\phi\phi} = r^2\sin^2\theta$

All others are zero e.g

$$\vec{g}_\phi \cdot \vec{g}_r = (-r\sin\theta\sin\phi\cos\phi) + (r\sin\theta\cos\phi\sin\phi)$$

$$= 0$$

Thus

- $$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$
- $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \, d\phi^2$

Since the metric is diagonal

$$g^{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix}$$

And thus

$$\vec{g}^r = g^{ra}\vec{g}_a = g^{rr}\vec{g}_r = \vec{g}_r$$

$$\vec{g}^\theta = g^{\theta\theta}\vec{g}_\theta = \frac{1}{r^2}\vec{g}_\theta$$

$$\vec{g}^\phi = \frac{\vec{g}_\phi}{r^2\sin^2\theta}$$

c) Then

$$dV = (\det g_{ab})^{1/2}\, du^1\, du^2\, du^3$$

$$= (1 \cdot r^2 \cdot r^2\sin^2\theta)^{1/2}\, dr\, d\theta\, d\phi$$

$$= r^2\sin\theta\, dr\, d\theta\, d\phi$$

d) Now we note by definition

- $\partial_a \vec{g}_b = \Gamma_{ab}^{\ c} \vec{g}_c$ and recall that $\Gamma_{ab}^{c} = \Gamma_{ba}^{\ c}$

Thus we have to compute:

$$\partial_r \vec{g}_r = 0$$

$$\partial_r \vec{g}_\theta \ , \ \partial_r \vec{g}_\phi$$

$$\partial_\theta \vec{g}_\theta \ , \ \partial_\theta \vec{g}_\phi$$

$$\partial_\phi \vec{g}_\phi$$

Working we have

- $\partial_r \vec{g}_\theta = \dfrac{\vec{g}_\theta}{r}$ • $\partial_r \vec{g}_\phi = \dfrac{g_\phi}{r}$, thus $\Gamma_{r\theta}^{\ \theta} = \dfrac{1}{r}$ $\Gamma_{r\phi}^{\ \phi} = \dfrac{1}{r}$

- $\partial_\theta \vec{g}_\theta = \dfrac{\partial}{\partial\theta} \left(+r\cos\theta\cos\phi , +r\cos\theta\sin\phi , -r\sin\theta \right)$

$$= \left(-r\sin\theta\cos\phi , -r\sin\theta\sin\phi , -r\cos\theta \right)$$

- $\partial_\theta \vec{g}_\theta = -r\vec{g}_r$

$$\Gamma_{\theta\theta}^{\ r} = -r$$

Then

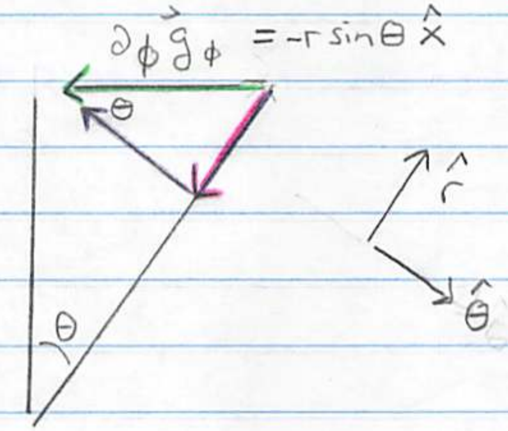
- $\partial_\theta \vec{g}_\phi = (-r\cos\theta \sin\phi, -r\cos\theta \cos\phi, 0)$

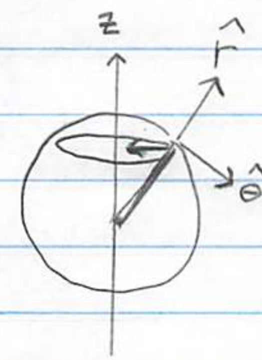
$$= \frac{\cos\theta}{\sin\theta} \vec{g}_\phi \qquad \Gamma^\phi_{\theta\phi} = \cot\theta$$

$$\partial_\phi \vec{g}_\phi = (-r\sin\theta\cos\phi, -r\sin\theta\sin\phi, 0)$$

Take $\phi = 0$

(☆)





From the picture $\partial_\phi \vec{g}_\phi$ is directed inward (but ⊥ to the z-axis) since it is like centripetal acceleration. We may decompose it into components directed in the $-\hat{r}$ direction ($\hat{r} = \vec{g}_r$) and the (negative) $\hat{\theta}$ direction ($\hat{\theta} = \vec{g}_\theta / r$). Thus

$$\partial_\phi \vec{g}_\phi = r\sin\theta\,(-\sin\theta\, \vec{g}_r - \cos\theta\, \vec{g}_\theta / r)$$

Picture!

One can check explicitly that this decomposition works. Thus

$$\Gamma^r_{\phi\phi} = -r\sin^2\theta \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

Summarizing

$$\begin{cases} \Gamma^{r}_{\theta\theta} = -r \\ \Gamma^{r}_{\phi\phi} = -r\sin^2\theta \end{cases}$$

$$\begin{cases} \Gamma^{\theta}_{r\theta} = \dfrac{1}{r} \\ \Gamma^{\theta}_{\phi\phi} = -\cos\theta\sin\theta \end{cases}$$

$$\begin{cases} \Gamma^{\phi}_{r\phi} = \dfrac{1}{r} \\ \Gamma^{\phi}_{\theta\phi} = \dfrac{\cos\theta}{\sin\theta} \end{cases}$$

A picture explaing $\Gamma^{r}_{\phi\phi}$ and $\Gamma^{\theta}_{\phi\phi}$ is given on the previous page.

e) Using the formula

- $\Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab})$

Then $g_{\phi\phi} = + r^2 \sin^2\theta$ while $g_{r\phi} = 0$ thus

- $\Gamma^r_{\phi\phi} = \frac{1}{2} g^{rr} (-\partial_r g_{\phi\phi})$, $g^{rr} = 1$, and so

$\Gamma^r_{\phi\phi} = -\sin^2\theta.$

Similarly

- $\Gamma^\theta_{\phi\phi} = \frac{1}{2} g^{\theta\theta} (-\partial_\theta g_{\phi\phi}) = \frac{1}{2}\left(\frac{1}{r^2}\right)\left(-\partial_\theta (r^2 \sin^2\theta)\right)$

$= - \sin\theta \cos\theta$

(f) Then

$$\int \frac{d\phi}{2\pi} \cos\phi \, \hat{\rho} \neq 0$$

because $\hat{\rho} = \cos\phi \, \hat{x} + \sin\phi \, \hat{y}$ depends on space, yielding

$$\int \frac{d\phi}{2\pi} \cos\phi \left[(\cos\phi)\,\hat{x} + (\sin\phi)\,\hat{y} \right] = \frac{1}{2}\hat{x} .$$

we used that $\langle \cos^2 \rangle = \langle \sin^2 \rangle = \frac{1}{2}$ to evaluate the integral. We also used $\langle \cos \sin \rangle = 0$

g) $\nabla \times \vec{A} = \vec{e}_i \, \varepsilon^{ijk} \, \partial_j A_k$

We replace $\partial_j = \nabla_j$ in cartesian coordinates. Then inserting identity, $\delta^i_{i'} = (M_0^{-1})^i{}_a (M_0)^a{}_{i'}$, and similarly for j, k we find:

$$\nabla \times \vec{A} = e_i (M_0^{-1})^i{}_a (M)^a{}_{i'} \times$$

$$(\nabla_j A_k)(M_0^{-1})^j{}_b (M_0)^b{}_{j'} (M_0^{-1})^k{}_c (M_0)^c{}_{k'} \times$$

$$\varepsilon^{i'j'k'}$$

$$= \vec{g}_a (\nabla_b A_c) \, \varepsilon^{abc}$$

We used for instance the tensorial property:

$$\nabla_b A_c = (\nabla_i A_j)(M_0^{-1})^i{}_b (M_0^{-1})^j{}_c$$

And

$$\vec{g}_a = \vec{e}_i (M_0^{-1})^i{}_a \qquad \varepsilon^{abc} = (M\,M\,M)\,\varepsilon^{ijk}$$

Here

$$(M_0)^a{}_i = \frac{\partial u^a}{\partial x^i} \qquad (M_0^{-1})^i{}_a = \frac{\partial x^i}{\partial u^a}$$

Since

- $\nabla \times \vec{A} = \vec{g}_a \, \varepsilon^{abc} \, \nabla_b A_c$

We note

- $\varepsilon^{abc} \nabla_b A_c = \varepsilon^{abc} (\partial_b A_c - \Gamma^{d}_{bc} A_d).$

But note in the second term:

- $\varepsilon^{abc} \Gamma^{d}_{bc} = 0$ since ε^{abc} is odd under interchange of (b,c), while Γ^{d}_{bc} is even under interchange of (b,c). Thus

$$(\nabla \times \vec{A}) = \vec{g}_a \, \varepsilon^{abc} (\partial_b A_c)$$

Using for orthogonal systems $\vec{g}_a = h_a \vec{e}_{\hat{a}}$, and $A_c = h_c A_{\hat{c}} = h_c A^{\hat{c}}$, and the fact that

$$\varepsilon^{abc} = \frac{1}{h_1 h_2 h_3} [abc],$$

we determine

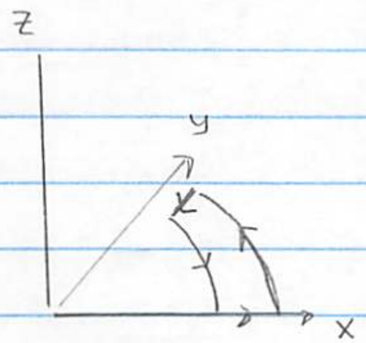
$$(\nabla \times \vec{A}) = \frac{1}{h_2 h_3} \vec{e}_{\hat{1}} \left(\frac{\partial}{\partial u^2} (h_3 A^{\hat{3}}) - \frac{\partial}{\partial u^3} (h_2 A^{\hat{2}}) \right)$$

$$+ \frac{1}{h_1 h_3} \vec{e}_{\hat{2}} \left(\frac{\partial}{\partial u^3} (h_1 A^{\hat{1}}) - \frac{\partial}{\partial u^1} (h_3 A^{\hat{3}}) \right) + \text{last term}$$

h) The stokes theorem says

$$\int d\vec{a} \cdot (\nabla \times \vec{V}) = \oint d\vec{l} \cdot \vec{V}$$

Then $da = r\, dr\, d\theta = h_r h_\theta\, dr\, d\theta \qquad h_r = 1 \quad h_\theta = r$



in $h_1 h_2 du^1 du^2$ is the area of the 1,2 plane.

$$d\vec{a} = h_r h_\theta\, dr\, d\theta\, (-\hat{\phi})$$

So $\qquad \hat{\phi}$ component of $\nabla \times \vec{V}$

$$d\vec{a} \cdot \nabla \times \vec{V} = -h_r h_\theta\, dr\, d\theta\, (\nabla \times \vec{V})_\phi$$

Then

$$(\nabla \times \vec{V})_\phi = \frac{1}{h_r h_\theta}\left[\frac{\partial(h_\theta V^{\hat{\theta}})}{\partial r} - \frac{\partial(h_r V^{\hat{r}})}{\partial \theta}\right]$$

So

$$\int d\vec{a} \cdot (\nabla \times \vec{V}) = -\left(\frac{\partial(h_\theta V^{\hat{\theta}})}{\partial r} - \frac{\partial(h_r V^{\hat{r}})}{\partial \theta}\right) dr\, d\theta$$

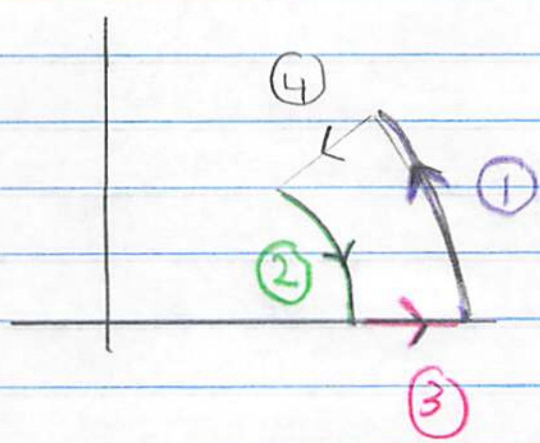
Integrating

$$\int d\vec{a}\cdot(\nabla\times\vec{V}) = -\int_{r_1}^{r_2} dr \int_{\theta_1}^{\theta_2} d\theta \left(\frac{\partial (h_\theta V^{\hat{\theta}})}{\partial r} - \frac{\partial (h_r V^{\hat{r}})}{\partial \theta} \right)$$

$$= \int_{\theta_1}^{\theta_2} d\theta \left[-h_\theta V^{\hat{\theta}} \Big|_{r=r_2} + h_\theta V^{\hat{\theta}} \Big|_{r=r_1} \right]$$

$$+ \int_{r_1}^{r_2} dr \left[h_r V^{\hat{r}} \Big|_{\theta=\theta_1} - h_r V^{\hat{r}} \Big|_{\theta=\theta_2} \right]$$

Each of four underlined terms is the contribution of one leg of the line integral for the loop shown



Take leg #1 :

$$d\vec{l}_{①} = \underbrace{h_\theta d\theta}_{\text{length per } d\theta} \overbrace{(-\hat{\theta})}^{\text{direction}} \qquad d\vec{l}_{①}\cdot\vec{V} = -h_\theta d\theta\, (\hat{\theta}\cdot\vec{V})$$

$$= -h_\theta d\theta\, V^{\theta}$$

Thus

$$\int d\vec{a} \cdot (\nabla \times \vec{V}) = \int_{1} d\vec{l}_{1} \cdot \vec{V} + \int_{2} d\vec{l}_{2} \cdot \vec{V} + \int_{3} d\vec{l}_{3} \cdot \vec{V}$$

$$+ \int_{4} d\vec{l}_{4} \cdot \vec{V}$$

$$= \oint d\vec{l} \cdot \vec{V}$$

i) Divergence

$$\nabla \cdot \vec{V} = \nabla_a V^a = \nabla_r V^r + \nabla_\theta V^\theta + \nabla_\phi V^\phi$$

$$= \partial_a V^a + \Gamma^a_{ab} V^b$$

Using the diagonal Christofell's

$$\Gamma^\theta_{\theta r} = \frac{1}{r} \qquad \Gamma^\phi_{\phi r} = \frac{1}{r} \qquad \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$$

We have

$$\nabla \cdot V = \underbrace{\left(\partial_r V^r\right)}_{\nabla_r V^r} + \underbrace{\left(\partial_\theta V^\theta + \frac{V^r}{r}\right)}_{\nabla_\theta V^\theta} + \underbrace{\left(\partial_\phi V^\phi + \frac{V^r}{r} + \cot\theta V^\theta\right)}_{\nabla_\phi V^\phi}$$

So regrouping

$$\nabla \cdot V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V^r) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\, V^\theta) + \frac{\partial V^\phi}{\partial \phi}$$

This agrees with the other expression we had:

$$\nabla \cdot V = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}\, V^a) = \frac{1}{r^2 \cancel{\sin\theta}}\partial_r(r^2 \cancel{\sin\theta}\, V^r) + \frac{1}{\cancel{r^2}\sin\theta}\partial_\theta(\cancel{r^2}\sin\theta\, V^\theta)$$

$$+ \frac{1}{\cancel{r^2}\cancel{\sin\theta}}\partial_\phi(\cancel{r^2}\cancel{\sin\theta}\, V^\phi)$$

Finally note

$$v^{\hat{r}} = v^r \qquad v^{\hat{\theta}} = \overset{r}{h_\theta}\, v^\theta \qquad v^{\hat{\phi}} = \overset{r\sin\theta}{h_\phi}\, v^\phi$$

Yielding

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v^r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v^{\hat{\theta}}) + \frac{1}{r\sin\theta}\partial_\phi v^{\hat{\phi}}$$

which agrees with the textbook formula.

Finally

$$T^{\hat{r}\hat{\theta}} = h_r h_\theta \, T^{r\theta}$$

$$= h_r h_\theta \left(\nabla^r v^\theta + \nabla^\theta v^r - \frac{2}{3} \cancelto{0}{g^{r\theta}} \, \nabla \cdot v \right)$$

$$= h_r h_\theta \left[\frac{1}{h_r^2} (\nabla_r v^\theta) + \frac{1}{h_\theta^2} (\nabla_\theta v^r) \right]$$

$$= h_\theta \left(\partial_r v^\theta + \Gamma^\theta_{\theta r} v^\theta \right) + \frac{1}{h_\theta} \left(\partial_\theta v^r + \Gamma^r_{\theta\theta} v^\theta \right)$$

$$= r \left(\partial_r (v^{\hat{\theta}}/r) + \frac{1}{r^2} v^{\hat{\theta}} \right) + \frac{1}{r} \left(\partial_\theta v^{\hat{r}} - r \frac{v^{\hat{\theta}}}{r} \right)$$

$$T^{\hat{r}\hat{\theta}} = \partial_r v^{\hat{\theta}} + \frac{1}{r} \partial_\theta v^{\hat{r}} - \frac{v^{\hat{\theta}}}{r}$$

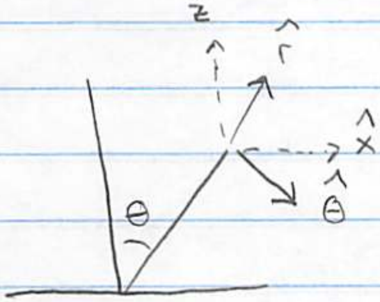
If $T^{\hat{r}\hat{\theta}} = T^{\hat{\theta}\hat{r}}$ is the only component then:

$$\overleftrightarrow{T} = T^{\hat{r}\hat{\theta}} \left(\vec{e}_r \vec{e}_\theta + \vec{e}_\theta \vec{e}_r \right)$$

$\swarrow$ you are not allowed to commute these: $\vec{e}_r \otimes \vec{e}_\theta \neq \vec{e}_\theta \otimes \vec{e}_r$

It is a tensor product here!

Then the picture is the following:



$$\hat{\theta} = -\sin\theta \, \hat{z} + \cos\theta \, \hat{x} \equiv \vec{e}_\theta$$

$$\hat{r} = \cos\theta \, \hat{z} + \sin\theta \, \hat{x} \equiv \vec{e}_r$$

Thus $\vec{e}_r, \vec{e}_\theta$ are just rotated versions of $\hat{x}$ and $\hat{z}$

So

for $\boldsymbol{e}_\theta \otimes \boldsymbol{e}_r$ you need to exchange the order of the two terms

This is $e_r \otimes e_\theta$

$$\overleftrightarrow{T} = T^{\hat{r}\hat{\theta}} \left[(\cos\theta \, \hat{z} + \sin\theta \, \hat{x})(-\sin\theta \, \hat{z} + \cos\theta \, \hat{x}) + \vec{e}_\theta \vec{e}_r \right]$$

$$= T^{\hat{r}\hat{\theta}} \left[-\sin 2\theta \, \hat{z}\hat{z} + \sin 2\theta \, \hat{x}\hat{x} + \cos 2\theta \, (\hat{x}\hat{z} + \hat{z}\hat{x}) \right]$$

Comparison with the cartesian form $\overleftrightarrow{T} = T^{ij} \vec{e}_i \vec{e}_j$

$$\overleftrightarrow{T} = T^{xx} \hat{x}\hat{x} + T^{xz} \hat{x}\hat{z} + T^{zx} \hat{z}\hat{x} + T^{zz} \hat{z}\hat{z}$$

Gives

$$T^{xx} = T^{\hat{r}\hat{\theta}} \sin 2\theta$$

$$T^{zz} = -T^{\hat{r}\hat{\theta}} \sin 2\theta$$

$$T^{xz} = T^{zx} = T^{\hat{r}\hat{\theta}} \cos 2\theta$$

This is just a rotation by angle θ (see picture above!)

$$T^{\hat{r}\hat{\theta}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow T^{\hat{r}\hat{\theta}} \begin{pmatrix} +\sin 2\theta & \cos 2\theta \\ \cos 2\theta & -\sin 2\theta \end{pmatrix}$$

From the local cartesian basis $\vec{e}_r, \vec{e}_\theta$ to the global cartesian basis $\hat{x}, \hat{y}$. Compare with the previous homework.