Problem 1. Practice with delta-fcns

A delta-function is a infinitely narrow spike with unit integral. $\int dx\, \delta(x) = 1$.

(a) Show that

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{1}$$

(b) Using the identity of part (b), show that

$$\delta(g(x)) = \sum_m \frac{1}{|g'(x_m)|}\delta(x - x_m) \qquad \text{where } g(x_m) = 0 \text{ and } g'_m(x_m) \neq 0 \tag{2}$$

(c) Show that

$$\int_0^\infty dx\, \delta(\cos(x))\, e^{-x} = \frac{1}{2\sinh(\pi/2)} \tag{3}$$

The delta function $\delta(x)$ should be thought of as sequence of functions $\delta_\epsilon(x)$ – known as a Dirac sequence – which becomes infinitely narrow and have integral one. For example, an infinitely narrow sequence of normalized Gaussians

$$\delta(x) = \lim_{\epsilon\to 0} \delta_\epsilon(x) = \lim_{\epsilon\to 0} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{x^2}{2\epsilon^2}}\,. \tag{4}$$

The important properties are

$$1 = \int dx\, \delta_\epsilon(x) \tag{5}$$

and the convolution property

$$f(x) = \lim_{\epsilon\to 0} \int dx_o f(x_o)\delta_\epsilon(x - x_o) \tag{6}$$

I will notate any Dirac sequence with $\delta_\epsilon(x)$.

Delta functions are perhaps best thought about in Fourier space. In particular think about Eq. (6) in Fourier space. At finite epsilon this reads

$$f(k) \simeq f(k)\delta_\epsilon(k)\,. \tag{7}$$

So the Fourier transform of a Dirac sequence $\delta_\epsilon(k)$ should be essentially one, except at large k where the function $f(k)$ is presumably small.

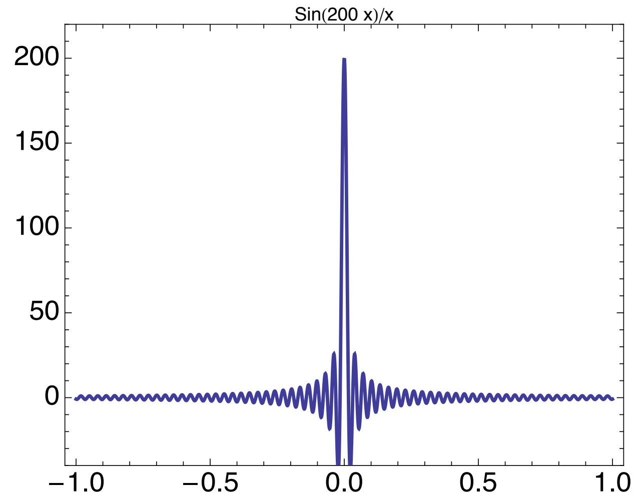
According to the uncertainty principle, a spike that has width $\Delta x \sim \epsilon$ in coordinate space, will have width $\Delta k \sim 1/\epsilon$ in k-space (momentum space). The meaningless formal expression

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} = \delta(x) \tag{8}$$

means that one should regulate this integral in some way and take the limit as the regulator ϵ goes to zero. For example, one could cut off the upper limit at a $k_{\max} = 1/\epsilon$,

$$\delta_\epsilon(x) = \int_{-1/\epsilon}^{1/\epsilon} \frac{dk}{2\pi}\, e^{ikx} = \frac{\sin(x/\epsilon)}{\pi x} \tag{9}$$

Making a graph of this function (with $1/\epsilon = 200$):



we see that for small ϵ it is infinitely narrow spike. Integrate around this spike between $-\Delta \ldots \Delta$, where Δ is small compared to one $\Delta \ll 1$, but much greater than ϵ, $\Delta \gg \epsilon$

$$I_\epsilon = \int_{-\Delta}^{\Delta} \mathrm{d}x \, \frac{\sin(x/\epsilon)}{\pi x} \tag{10}$$

$$= \int_{-\Delta/\epsilon}^{\Delta/\epsilon} \mathrm{d}u \, \frac{\sin(u)}{(\pi u)} \tag{11}$$

$$\simeq \int_{-\infty}^{\infty} \mathrm{d}u \, \frac{\sin(u)}{(\pi u)} \tag{12}$$

$$\simeq 1 \tag{13}$$

In the last steps we extended the integration to ∞ (since $\Delta/\epsilon \gg 1$), and have used the table integral, $\int_{-\infty}^{\infty} du \, \sin(u)/(\pi u) = 1$. The approximation becomes exact in the limit $\epsilon \to 0$, and thus we have shown that

$$\delta(x) = \lim_{\epsilon \to 0} \delta_\epsilon(x) = \lim_{\epsilon \to 0} \frac{\sin(x/\epsilon)}{\pi x} \tag{14}$$

is a Dirac sequence.

The precise way in which you regulate the Fourier integral is unimportant. The next problem regulates the Fourier integral in a particularly common way.

(d) Consider the Fourier transform pair $f(x)$ and $f(k) = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{ikx} f(x)$. Note that

$$f(k = 0) = \int_{-\infty}^{\infty} dx f(x) \tag{15}$$

Without using Mathematica, compute the following Fourier transform

$$\delta_\epsilon(x) \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, e^{ikx} e^{-\epsilon|k|} \tag{16}$$

You can check your algebra by explicitly checking that $\int dx \, \delta_\epsilon(x) = 1$ by direct integration. Explain to yourself why one knows this integral must be unity before doing the integral.

Verify that

$$\lim_{\epsilon \to 0} \delta_\epsilon(x) = \delta(x) \tag{17}$$

i.e. that $\delta_\epsilon(k)$ is a Dirac sequence. This is another proof that

$$\delta(x) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{ikx} e^{-\epsilon|x|} = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{ikx} \tag{18}$$

Problem 2. 3d delta-functions

A delta-function in 3 dimensions $\delta^3(\boldsymbol{r}-\boldsymbol{r}_o)$ is an infinitely narrow spike at $\boldsymbol{r}_o$ which satisfies

$$\int d^3\boldsymbol{r}\, \delta^3(\boldsymbol{r}-\boldsymbol{r}_o) = 1 \tag{19}$$

In spherical coordinates, where the measure is

$$d^3\boldsymbol{r} = r^2 dr\, d(\cos\theta)\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi\,, \tag{20}$$

we must have

$$\delta^3(\boldsymbol{r}-\boldsymbol{r}_o) = \frac{1}{r^2}\delta(r-r_o)\delta(\cos\theta-\cos\theta_o)\delta(\phi-\phi_o) = \frac{1}{r^2\sin\theta}\delta(r-r_o)\;\delta(\theta-\theta_o)\delta(\phi-\phi_o) \tag{21}$$

so that $\int d^3\boldsymbol{r}\,\delta^3(\boldsymbol{r}) = 1$. For a general curvilinear coordinate system

$$\delta^3(\boldsymbol{r}-\boldsymbol{r}_o) = \frac{1}{\sqrt{g}}\prod_a \delta(u^a - u^a_o) \tag{22}$$

where u^a_o are the coordinates of $\boldsymbol{r}_o$ and $\sqrt{g} = \left|\left|\frac{\partial(x^1,x^2,x^3)}{\partial(u^1,u^2,u^3)}\right|\right|$ is the appropriate (absolute value) of the jacobian determinant:

$$\left|\frac{\partial(x^1,x^2,x^3)}{\partial(u^1,u^2,u^3)}\right| = \det\begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{pmatrix} \tag{23}$$

(a) Working in a general coordinate system, show that

$$\int d^3\boldsymbol{r}\delta^3(\boldsymbol{r}-\boldsymbol{r}_0) = 1 \tag{24}$$

(b) What is formula $\delta^3(\boldsymbol{r}-\boldsymbol{r}_o)$ for cylindrical coordinates?

(c) A uniform ring of charge Q and radius a sits at height z_o above the xy plane, and the plane of the ring is parallel to the xy plane. Express the charge density $\rho(\boldsymbol{r})$ (charge per volume) in spherical coordinates using delta-functions. Check that the volume integral of $\rho(\boldsymbol{r})$ gives the total Q.

(d) Consider a charge Q spread uniformly over a flat circular disc of negligble thickness and radius R, lying flat in the xy plane at $z=0$. Express the charge density $\rho(r,\theta,\phi)$ in spherical coordinates using δ-fcns and θ functions.

Problem 3. Periodic pulses

A pulse of electric field takes the form

$$E_1(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/(2\sigma^2)} e^{-i\omega_o t} \tag{25}$$

where $\omega_o \gg \frac{1}{\sigma}$

(a) Show by direct integration that its Fourier transform is [1]

$$E_1(\omega) = e^{-\frac{1}{2}\sigma^2(\omega-\omega_o)^2} \tag{28}$$

Sketch $\mathrm{Re}E_1(t)$ and its power spectrum $|E_1(\omega)|^2$ qualitatively, paying heed to the parameters, $\omega_o \gg \frac{1}{\sigma}$.

(b) Suppose that the wave form repeats once, with the second pulse arriving at a time $\mathcal{T}_o$ after the first pulse. The total electric field, $E_2(t)$, consists of the first pulse $E_1(t)$ at centered at time $t = 0$, and a second indentical pulse at time $\mathcal{T}_o$, $E_2(t) = E_1(t) + E_1(t - \mathcal{T}_o)$. Show that the Fourier transform and the power spectrum is

$$E_2(\omega) = E_1(\omega)\left(1 + e^{i\omega\mathcal{T}_o}\right) \qquad |E_2(\omega)|^2 = |E_1(\omega)|^2 \left(2 + 2\cos(\omega\mathcal{T}_o)\right) \tag{29}$$

(c) Now suppose that we have n (with n odd) arranged almost symmetrically around $t = 0$, *i.e.*

$$E_n(t) = E_1(t+(n-1)\mathcal{T}_o/2)+\ldots+E_1(t+\mathcal{T}_o)+E_1(t)+E_1(t-\mathcal{T}_o)+\ldots E_1(t-(n-1)\mathcal{T}_o/2)\,, \tag{30}$$

so that for $n = 3$

$$E_3(t) = E_1(t + \mathcal{T}_o) + E_1(t) + E_1(t - \mathcal{T}_o)\,. \tag{31}$$

Show that

$$E_n(\omega) = E_1(\omega)\,\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)} \tag{32}$$

and

$$|E_n(\omega)|^2 = |E_1(\omega)|^2 \left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)}\right)^2 \tag{33}$$

These functions are shown below:

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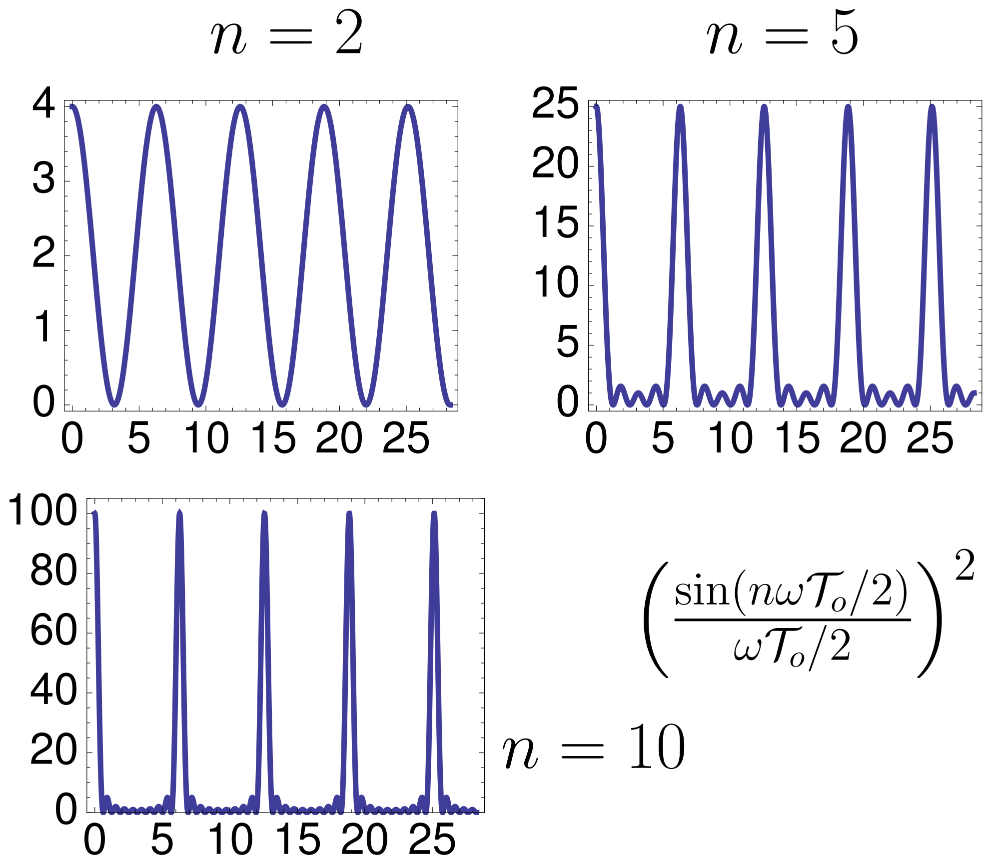
[1] As is conventional the Fourier transform of temporal signal is

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega) \tag{26}$$

$$f(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} f(t) \tag{27}$$

This has the opposite sign from the spatial fourier transform $\hat{f}(k) = \int dx e^{-ikx} f(x)$.


$n = 2$
$n = 5$
$n = 10$
$\left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\omega\mathcal{T}_o/2}\right)^2$
0 5 10 15 20 25
0 1 2 3 4
0 5 10 15 20 25
0 20 40 60 80 100

(d) By taking limits of your expressions in the previous part show that after n pulses, with $n \to \infty$, show that the sequence of functions given by Eq. (32) and Eq. (33) tends to the Dirac sequences

$$\lim_{n\to\infty} E_n(\omega) = \sum_m E_1(\omega_m) \frac{2\pi}{\mathcal{T}_o} \delta(\omega - \omega_m) \tag{34}$$

and

$$\lim_{n\to\infty} |E_n(\omega)|^2 = \underbrace{n\mathcal{T}_o}_{\text{total time}} \times \sum_m |E_1(\omega_m)|^2 \frac{2\pi}{\mathcal{T}_o^2} \delta(\omega - \omega_m) \tag{35}$$

where $\omega_m = 2\pi m/\mathcal{T}_o$.

(e) Qualitatively sketch the time-averaged power spectrum of part (d), *i.e.* sketch

$$\frac{\lim_{n\to\infty} |E_n(\omega)|^2}{\text{total time}}. \tag{36}$$

Compare your results for the time-averaged spectrum to the single pulse spectrum of part (a).

Remark We have in effect shown that if we define

$$\Delta(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t - n\mathcal{T}_o)\,. \tag{37}$$

Then the Fourier transform of $\Delta(t)$ is

$$\hat{\Delta}(\omega) = \sum_n e^{-i\omega n\mathcal{T}_o} = \sum_m \frac{2\pi}{\mathcal{T}_o} \delta(\omega - \omega_m)\,. \tag{38}$$

We used this to prove the Poisson summation formula in class

$$\sum_n f(n\mathcal{T}_o) = \frac{1}{\mathcal{T}_o} \sum_m \hat{f}(\omega_m) \tag{39}$$

where $f(t)$ is any function and $\hat{f}(\omega)$ is its Fourier transform.

Problem 4. Hankel Transforms and Bessel functions

Consider a 2D function $f(x, y) = f(r, \phi_r)$ and extend the Fourier transform in an obvious way

$$\hat{f}(\boldsymbol{k}) = \int d^2 r e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} f(\boldsymbol{r}) \tag{40}$$

$$f(\boldsymbol{r}) = \int \frac{d^2\boldsymbol{k}}{(2\pi)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \hat{f}(\boldsymbol{k}) \tag{41}$$

We use a vector notation here:

$$\boldsymbol{r} \equiv (x, y) = (r\cos\phi_r, r\sin\phi_r) \qquad \int d^2 r \equiv \int_{-\infty}^{\infty} dx \int_{\infty}^{\infty} dy = \int_0^{\infty} r dr \int_0^{2\pi} d\phi_r \tag{42}$$

$$\boldsymbol{k} \equiv (k_x, k_y) = (k\cos\phi_k, k\sin\phi_k) \qquad \int d^2 k \equiv \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y = \int_0^{\infty} k dk \int_0^{2\pi} d\phi_k \tag{43}$$

The exponential factor can be written in any number of ways

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \equiv e^{ik_x x + ik_y y} = e^{ikr\cos(\phi_k - \phi_r)} \tag{44}$$

Clearly at any fixed radius of magnitude r, the azimuthal dependence of the function $f(r, \phi_r)$ can be expanded in fourier series

$$f(r, \phi_r) = \frac{1}{2\pi} \sum_n f_n(r) e^{in\phi_r}\,. \tag{45}$$

where we have defined $\boldsymbol{r} = (r\cos\phi_r, r\sin\phi_r)$. Similarly, at any fixed wavenumber magnitude we may expand $f(k, \phi)_k$ in a Fourier series

$$\hat{f}(k, \phi_k) = \frac{1}{2\pi} \sum_n \hat{f}_n(k) e^{in\phi_k} \tag{46}$$

were we have defined $\boldsymbol{k} = (k_x, k_y) = (k\cos\phi_k, k\sin\phi_k)$. The relationship between $f_n(r)$ and $\hat{f}_n(k)$ is given by Hankel transforms which we develop below.

(a) Examine

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = e^{ikr\cos(\Delta\phi)} \tag{47}$$

where $\Delta\phi = (\phi_k - \phi_r)$ is the angle between the $\boldsymbol{r}$ and $\boldsymbol{k}$. Expand Eq. (47) in a power series in kr to at least fourth order, and collect it in powers of $e^{i\Delta\phi}$. (Hint: write Eq. (47) as $\exp(i\frac{kr}{2}e^{i\Delta\phi})\exp(i\frac{kr}{2}e^{-i\Delta\phi})$ before expanding.) You should find

$$\begin{aligned} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = &\left(\frac{1}{0!0!} - \frac{1}{1!1!}u^2 + \frac{1}{2!2!}u^4\right) + ie^{i\Delta\phi}\left(\frac{1}{0!1!}u - \frac{1}{1!2!}u^3\right) \\ &+ i^2 e^{i2\Delta\phi}\left(\frac{1}{0!2!}u^2 - \frac{1}{1!3!}u^4\right) + i^3 e^{i3\Delta\phi}\left(\frac{1}{0!3!}u^3\right) + i^4 e^{i4\Delta\phi}\left(\frac{1}{0!4!}u^4\right) \\ &+ \text{similar results for } e^{-in\Delta\phi} \text{ terms} \end{aligned} \tag{48}$$

with $u = kr/2$.

(b) Continuing the expansion of part (a), show that the fourier series assocated with $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ takes the form

$$e^{ikr\cos(\Delta\phi)} = \sum_{n=-\infty}^{\infty} J_n(kr) i^n e^{in\Delta\phi} \tag{49}$$

where for positive n

$$J_n(x) \equiv \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{2k} \tag{50}$$

and for negative n we just have defined

$$J_{-n}(x) = (-1)^n J_n(x) \tag{51}$$

$J_n(x)$ is the Bessel function of order n. Sketch the first four Bessel functions using Mathematica.

(c) Given the fundamental identity, Eq. (49), show that

$$\hat{f}_n(k) = 2\pi \int_0^{\infty} r dr \,(-i)^n J_n(kr) f_n(r) \tag{52}$$

$$f_n(r) = \int_0^{\infty} \frac{kdk}{2\pi}\, i^n J_n(kr) \hat{f}_n(k) \tag{53}$$

We say that $\hat{f}_n(k)$ is the *Hankel* transform of $f_n(r)$ of degree n.

(d) The completeness relation in 2D reads

$$\int \frac{d^2k}{(2\pi)^2} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} = \delta^2(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{r}\delta(r-r')\delta(\phi_r-\phi_r') \tag{54}$$

The fourier series are complete as well (see class notes):

$$\frac{1}{2\pi}\sum_n e^{in(\phi_r-\phi_r')} = \delta(\phi_r-\phi_r') \tag{55}$$

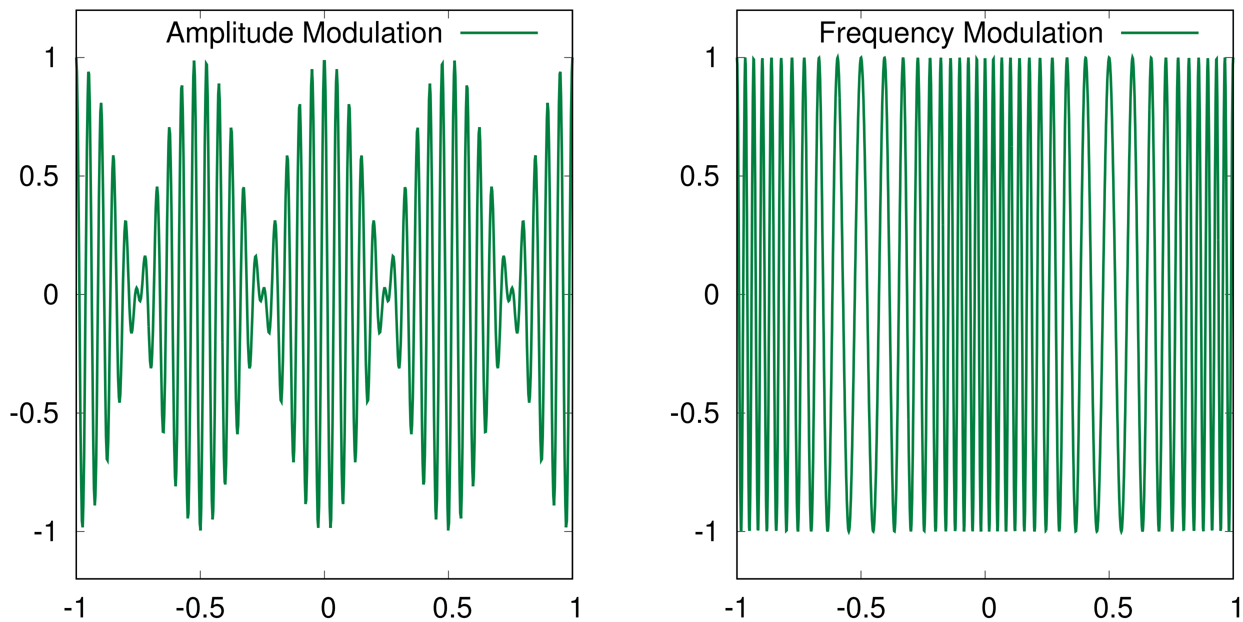
Show that Eq. (54) together with the fundamental identity Eq. (49) and Eq. (55) implies that $J_n(kr)$ are complete in the following sense:

$$\int_0^{\infty} kdk\, J_n(kr) J_n(kr') = \frac{1}{r}\delta(r-r')\,. \tag{56}$$

Use this to show that Eq. (52) is consistent with Eq. (53).

Problem 5. AM and FM

In radio transmission the "carrier" frequency is a high frequency compared to the signal. For example, the 820 in the radio station `AM 820 WNYC` stands for 820 kHz, while the 93.9 in the radio station `FM 93.9 WNYC` stands for 93.9 MHz. The signal frequency (i.e. sound) is much lower and usually measured in hundreds of Herz. (The "middle A" tuning note is 440 Hz). AM modulation encodes the signal by changing the amplitude of wave according to the signal frequency. FM modulation encodes the signal by slightly modifying the frequency of the carrier wave. AM modulation was considered in class. This problem considers FM modulation



Consider the FM modulated periodic function of time with period T_0:

$$S(t) = Ae^{-i(\omega_c t + \epsilon \cos(\omega_o t))} \tag{57}$$

- Here the carrier frequency is $\omega_c = n_c \frac{2\pi}{T_0}$ with $n_c \gg 1$.
- The signal frequency is $\omega_o = \frac{2\pi}{T_0}$.
- ϵ is the modulation parameter and controls the bandwidth of the signal. Indeed, the local frequency, defined as the time-derivative of the phase $\phi(t) = \omega_c t + \epsilon \cos(\omega_0 t)$,

$$\text{local frequency} \equiv \frac{d\phi(t)}{dt} \tag{58}$$

  varies between

$$\omega_c - \epsilon\omega_o \quad \ldots \quad \omega_c + \epsilon\omega_0. \tag{59}$$

- For $\omega_c = 20\omega_0$ and $\epsilon = 10$ a plot of the real part of this signal is shown above.

(a) Expand $S(t)$ as a Fourier series

$$S(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} S_n e^{-i\omega_n t} \tag{60}$$

You should find

$$\frac{S_n}{T_0 A} = i^{n-n_c} J_{n-n_c}(\epsilon) \tag{61}$$

(b) Plot the power spectrum:

$$\frac{|S_n|^2}{T_0^2 A^2} \tag{62}$$

with Mathematica. Take a carrier frequency $\omega_c = 820\omega_o$ and a modulation parameter of $\epsilon = 1$, and plot it between $\omega_n = 810\,\omega_o$ and $830\,\omega_o$. You should find that the signal is sharply peaked near $n = 820$. The relevant command is

```
DiscretePlot[BesselJ[n - 820, 1]^2, {n, 810, 830}, PlotRange->{0, 1.0}]
```

(i) Set ϵ to be a small number – say 0.1. What do you see? Why is this the required result? Explain

(ii) Set $\epsilon = 40$. What is the approximate frequency width of the power spectrum? You may need to change the plot domain and range to see the spectrum. What is the frequency width of the power spectrum in units of ω_o and why is this the expected result?

(iii) Keep $\epsilon = 40$ as in the last part. Use Mathematica to sum up the power in the frequency range from $\omega = 700\omega_o$ to $\omega = 900\,\omega_o$

```
Sum[BesselJ[n - 820, 40.0]^2, {n, 700, 900} ]
```

Explain this result.