Problem 1

a) $\delta(ax)$ clearly vanishes on every interval which excludes the origin

Then changing variables: $u = ax$

$$\int_{-\varepsilon}^{\varepsilon} dx\, \delta(ax) = \int_{-|a|\varepsilon}^{|a|\varepsilon} \frac{du}{|a|}\, \delta(u)$$

$$= \frac{1}{|a|}$$

So by definition

$$\delta(ax) = \frac{\delta(x)}{|a|}$$

b) The delta-fcn $\delta(g(x))$ can be non-zero only if $g(x_m) = 0$. Near a zero we have

$$g(x) = g'(x_m)(x - x_m)$$

So by the previous problem we have near x_m that $\delta(g(x)) = \delta(g'(x_m)(x - x_m))$ and thus

$$\delta(g(x)) = \sum_m \frac{1}{|g'(x_m)|}\, \delta(x - x_m)$$

$\boxed{c)}$ Then since $\cos x = 0$ for $x_m = \pi/2 + m\pi$

$$\delta(\cos(x)) = \sum_{m=-\infty}^{\infty} \frac{\delta(\pi/2 + m\pi)}{|\cos'(\pi/2 + m\pi)|}$$

Then

$$\left|\frac{d}{dx}\cos(x)\right| = \left|\,|\sin(x)|\,\right|_{x = \pi/2 + m\pi}$$

$$= 1$$

So then

$$\int_0^{\infty} dx\, \delta(\cos x)\, e^{-x} = \int_0^{\infty} dx\, e^{-x} \sum_{m=-\infty}^{\infty} \delta(\pi/2 + m\pi)$$

$$= \sum_{m=0}^{\infty} e^{-(\pi/2 + m\pi)} = e^{-\pi/2}\left[1 + e^{-\pi} + e^{-2\pi} + \dots\right]$$

$$= \frac{e^{-\pi/2}}{1 - e^{-\pi}} = \frac{1}{e^{\pi/2} - e^{-\pi/2}} = \frac{1}{2\sinh(\pi/2)}$$

$\boxed{d)}$ $$\delta_\varepsilon(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, e^{-\varepsilon|k|} \qquad (\text{Eq.} \star)$$

$$= \int_0^{\infty} \frac{dk}{2\pi}\, e^{ik(x+i\varepsilon)} + \int_{-\infty}^{0} \frac{dk}{2\pi}\, e^{ik(x-i\varepsilon)}$$

$$= \frac{1}{2\pi}\left[\frac{-1}{i(x+i\varepsilon)} + \frac{1}{i(x-i\varepsilon)}\right]$$

$$\underline{\underline{\delta_\varepsilon(x) = \frac{1}{\pi}\,\frac{\varepsilon}{x^2+\varepsilon^2}}} \qquad (\text{Eq} \star\star)$$

- Then clearly this vanishes for $\varepsilon \to 0$ for all intervals not containg the origin. And, we have from Eq $\star$ and the theory of fourier transforms:

$$e^{-\varepsilon|k|} = \int_{-\infty}^{\infty} e^{-ikx}\, \delta_\varepsilon(x)\, dx .$$

- Setting $k=0$ we have

$$1 = \int_{-\infty}^{\infty} \delta_\varepsilon(x)\, dx ,$$

Which can be verified by direct integration of Eq $\star\star$. Thus $\delta_\varepsilon(x)$ is correctly normalized, so that

$$\lim_{\varepsilon \to 0} \frac{1}{\pi}\,\frac{\varepsilon}{x^2+\varepsilon^2} = \delta(x)$$

Problem 2

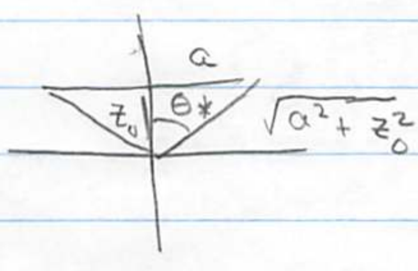
$\boxed{a)}$ $$\int d^3r\, \delta^3(\vec{r}-\vec{r}_0) = \int \cancel{\sqrt{g}}\, du^1 du^2 du^3 \times \frac{1}{\cancel{\sqrt{g}}} \delta(u^1-u^1_0)\delta(u^2-u^2_0) \times \delta(u^3-u^3_0)$$

$$= 1$$

$\boxed{b)}$ $$\delta^3(\vec{r}-\vec{r}_0) = \frac{1}{\rho}\delta(\rho-\rho_0)\delta(\phi-\phi_0)\delta(z-z_0)$$

$\boxed{c)}$ $$\rho(r) = \frac{1}{r^2 \sin\theta_*} \delta\left(r-\sqrt{a^2+z_0^2}\right)\delta(\theta-\theta_*)\frac{Q}{2\pi}$$

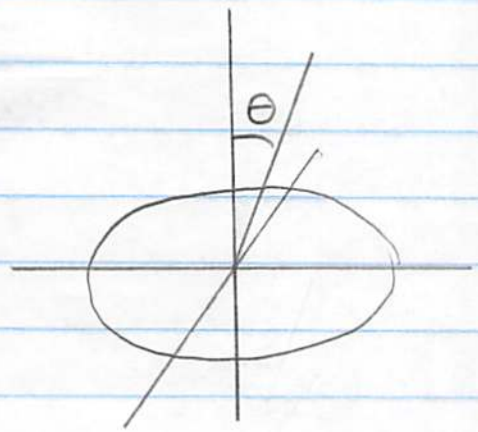
Here $\cos\theta_* = \dfrac{z_0}{(z_0^2+a^2)^{1/2}}$



- Clearly integrating gives Q

$$\int r^2 \sin\theta \, dr\, d\theta\, d\phi\, \rho(r) = Q$$

$\boxed{d)}$



- Then, $dz = r\sin\theta \, d\theta + \sin\theta \, dr$. Note the charge density has support only at $\theta = \pi/2$, where $dz = r \, d\theta$ and thus

$$\delta(z) = \frac{1}{r}\,\delta(\theta - \pi/2)$$

Yielding

$$\rho(r) = \frac{Q}{\pi R^2}\,\frac{1}{r}\,\delta(\theta - \pi/2)\,\Theta(R - r)$$

- You can check

$$\int r^2 \sin\theta \, dr \, d\theta \, d\phi \; \rho(r)$$

$$= \frac{Q}{\pi R^2} \int_0^R r \, dr \int_0^{2\pi} d\phi = Q$$

Problem 3

a) The

$$E(\omega) = \int_{-\infty}^{\infty} e^{+i\omega t} \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-t^2/2\sigma^2} e^{-i\omega_0 t} \, dt$$

- To do this integral first define $\bar{t} \equiv \frac{t}{\sigma}$ and $\bar{\omega} \equiv \sigma(\omega - \omega_0)$ which is motivated by units

$$E(\omega) = \int_{-\infty}^{\infty} \frac{d\bar{t}}{\sqrt{2\pi}} \, e^{-\bar{t}^2/2 + i\bar{\omega}\bar{t}}$$

- Now complete the square

$$u \equiv \bar{t} - i\bar{\omega}$$

$$u^2 = \bar{t}^2 - 2i\bar{\omega}\bar{t} + \bar{\omega}^2$$

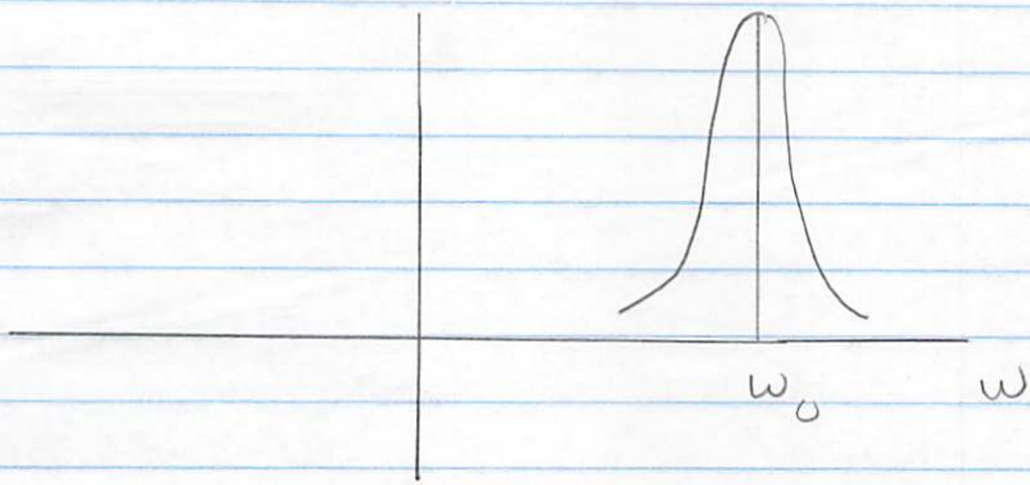
And

$$E(\omega) = e^{-\bar{\omega}^2/2} \int_{-\infty + i\bar{\omega}}^{\infty + i\bar{\omega}} du \, \frac{e^{-u^2/2}}{\sqrt{2\pi}} = e^{-\bar{\omega}^2/2} \int_{-\infty}^{\infty} du \, \frac{e^{-u^2}}{\sqrt{2\pi}}$$

requires Cauchy theorem

$$E(\omega) = e^{-\bar{\omega}^2/2}$$

$$= e^{-\sigma^2(\omega - \omega_0)^2/2}$$



b) Then for two pulses

$$E_2(\omega) \equiv \int_{-\infty}^{\infty} \left(E_1(t) + E_1(t - T_0) \right) e^{+i\omega t}$$

- $$E_2(\omega) = E_1(\omega) + E_1(\omega) e^{i\omega T_0}$$
- $$|E_2(\omega)|^2 = |E_1(\omega) + E_1(\omega) e^{i\omega T_0}|^2$$

$$= |E_1(\omega)|^2 \left(1 + e^{i\omega T_0} + e^{-i\omega T_0} + 1 \right)$$

$$= |E_1(\omega)|^2 \left(2 + 2\cos(\omega T_0) \right)$$

c) For n pulses we have following part b)

$$E_n(\omega) = E_1(\omega) e^{-i\omega(n-1)T_0/2} + E_1 e^{-i\omega(n-1)T_0/2} e^{i\omega T_0}$$

$$+ \ldots\ldots + E_1 e^{-i\omega(n-1)T_0/2} e^{i(n-1)\omega T_0}$$

↑ Take $n=3$ where

$$E_3(\omega) = E_1(\omega)\, e^{-i\omega T_0} + E_1(\omega) + E_1(\omega)\, e^{i\omega T_0}$$

etc. Now pulling out a common factor of $E_1\, e^{-i\omega(n-1)T_0/2}$, we have:

$$1+z+\dots z^n = \frac{(1-z^{n+1})}{1-z}$$

$$E_n(\omega) = E_1(\omega)\, e^{-i\omega(n-1)T_0/2}$$

$$\left[1 + e^{+i\omega T_0} + e^{2i\omega T_0} + \dots\ e^{(n-1)i\omega T_0} \right]$$

$$= E_1(\omega) \left[\frac{e^{-i\omega(n-1)T_0/2}\,(1 - e^{i\omega n T_0})}{1 - e^{i\omega T_0}} \right]$$

$$= E_1(\omega) \left[\frac{\sin(n\omega T_0/2)}{\sin(\omega T_0/2)} \right]$$

Then

$$|E_n(\omega)|^2 = E_1(\omega) \left[\frac{\sin(n\omega T_0/2)}{\sin(\omega T_0/2)} \right]^2$$

d) From the graphics we see that the functions have spikes at $\omega \simeq \frac{2\pi}{T_0} m$.

- We only need to show that these spikes are appropriately normalized. To this end integrate around a spike. Take the spike at $m=0$ for simplicity with n large

$$\boxed{I_1 = \int_{-\varepsilon}^{\varepsilon} \frac{\sin(n\omega T_0/2)}{\sin(\omega T_0/2)} d\omega}$$

- Since ω is small $[-\varepsilon, \dots \varepsilon]$ we may expand the denominator

$$\sin(\omega T_0/2) \simeq \omega T_0/2$$

Then

$$I_1 = \frac{2}{T_0} \int_{-\varepsilon}^{\varepsilon} \frac{d\omega}{\omega} \sin(n\omega T_0/2)$$

- But, we may not expand the numerator since n is arbitrarily large. Define $u = n\omega T_0/2$. Then

$$I_1 = \frac{2}{T_0} \int_{-\varepsilon n\omega T_0/2}^{\varepsilon n\omega T_0/2} \frac{du}{u} \sin(u)$$

Taking $n \to \infty$

$$\underbrace{I = \frac{2}{T_0} \int_{-\infty}^{\infty} \frac{du}{u} \sin(u)}$$

$$\boxed{I_1 = \frac{2\pi}{T_0}}$$

use mathematica or complex analysis π

Thus, we have shown

- $$\lim_{n\to\infty} E_n(\omega) = \sum_n E_1(\omega_n) \frac{2\pi}{T_0} \delta(\omega - \omega_m)$$

- Similarly we have to compute the integral:

$$I_2 \equiv \int_{-\varepsilon}^{\varepsilon} \left[\frac{\sin(n\omega T_0/2)}{\sin(\omega T_0/2)} \right]^2 d\omega$$

We have, following the same steps as before,

$$u \equiv n\omega T_0/2 \qquad \frac{d\omega}{\omega} = \frac{du}{u}$$

We have

$$I_2 = \frac{n}{(T_0/2)} \int_{-n\varepsilon T_0/2}^{n\varepsilon T_0/2} \left(\frac{\sin(u)}{u}\right)^2 du$$

Taking $n \to \infty$ and using the table integral

$$\int_{-\infty}^{\infty} du \left(\frac{\sin u}{u}\right)^2 = \pi$$

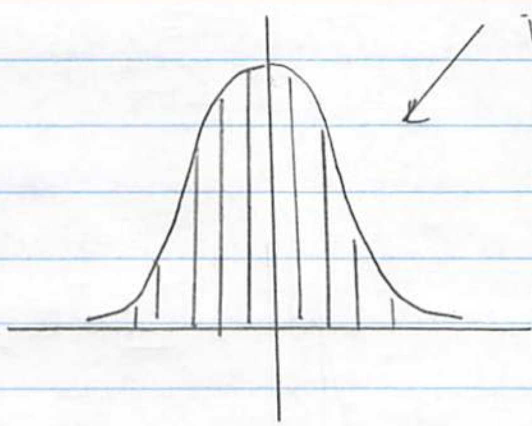
Gives

$$I_2 = n T_0 \frac{2\pi}{T_0^2}$$

Thus we have established the limit

$$\lim_{n\to\infty} \frac{|E_n(\omega)|^2}{\text{Total time}} = \sum_m |E_1(\omega_m)|^2 \frac{2\pi}{T_0^2} \delta(\omega - \omega_m)$$

$\boxed{e)}$ The picture is the following



instead of the original smooth curve one finds a set of spikes separated by $\frac{2\pi}{T_0} = \Delta\omega$

The relative magnitudes of the spikes is given by $E_1(\omega)$.

Problem 4

a) Take $\exp\left(\frac{ikr}{2} e^{i\Delta\phi}\right) \exp\left(\frac{ikr}{2} e^{-i\Delta\phi}\right) = e^{i\vec{k}\cdot\vec{r}}$

- The expansion then is a produt

$$e^{i\vec{k}\cdot\vec{r}} = \left[1 + \frac{i}{1!}\left(\frac{kr}{2}\right) e^{i\Delta\phi} + \frac{i^2}{2!}\left(\frac{kr}{2}\right)^2 e^{i2\Delta\phi} + \ldots\right]$$

$$\times \left[1 + \frac{i}{1!}\left(\frac{kr}{2}\right) e^{-i\Delta\phi} + \frac{(+i)^2}{2!}\left(\frac{kr}{2}\right)^2 e^{-i2\Delta\phi} + \ldots\right]$$

- Then for instance the way you get $(e^{i\Delta\phi})$ is by selecting $e^{in\Delta\phi}$ terms from the first sum in square brackets and $e^{-i(n-1)\Delta\phi}$ terms in the second bracket.

- Thus -

$$e^{i\vec{k}\cdot\vec{r}} = (e^{i0\phi})^0 \left[1\cdot 1 + \frac{i\cdot i}{1!\,1!}\left(\frac{kr}{2}\right)^2 + \frac{i^2}{2!}\frac{i^2}{2!}\left(\frac{kr}{4}\right)^2 + \ldots\right]$$

$$+ (e^{i\Delta\phi})\left[\frac{i}{1!}\left(\frac{kr}{2}\right) + \frac{i^2}{2!}\frac{i}{1!}\left(\frac{kr}{2}\right)^2\left(\frac{kr}{2}\right) + \ldots\right]$$

$$+ \ldots$$

$$+ e^{i2\Delta\phi}\left[\frac{i^2}{2!}\left(\frac{kr}{2}\right)^2 + \frac{i^3}{3!}\left(\frac{kr}{2}\right)^3 \frac{i}{1!}\left(\frac{kr}{2}\right) + \cdots\right]$$

$$+ e^{i3\Delta\phi}\left[\frac{i^3}{3!}\left(\frac{kr}{2}\right)^3\right]$$

$$+ e^{i4\Delta\phi}\left[\frac{i^4}{4!}\left(\frac{kr}{2}\right)^4\right] + \text{negative terms}$$

This agrees with the expression in the handout.

$\boxed{b)}$ From the discussion given abouve the coefficient of $e^{in\Delta\phi}$ is for $n>0$

$$C_n = \sum_{k=0} \frac{i^{n+k}}{(n+k!)} \frac{i^k}{k!}\left(\frac{kr}{2}\right)^{n+k}\left(\frac{kr}{2}\right)^k$$

$$= i^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!}\frac{1}{k!}\left(\frac{kr}{2}\right)^{n+2k}$$

$$C_n = i^n J_n(kr)$$

For $n<0$

$$C_n = \sum_{k=0}^{\infty} \frac{i^{|n|+k}}{(|n|+k)!}\frac{i^k}{k!}\left(\frac{kr}{2}\right)^{|n|+k}\left(\frac{kr}{2}\right)^k$$

So for $n<0$

$$C_n = i^{|n|} J_{|n|}(kr)$$

$$= (-1)^n i^n J_{|n|}(kr)$$

$$\equiv i^n J_n(kr) \quad \text{where for } n<0$$

$$J_n(kr) \equiv (-1)^{|n|} J_{|n|}(kr)$$

Putting together these expressions

$$e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\Delta\phi} = \sum_{n=-\infty}^{\infty} J_n(kr)\, i^n e^{in\Delta\phi}$$

$\boxed{c)}$ From the fourier integral

$$f(\vec{k}) = \int d^2r\, e^{-i\vec{k}\cdot\vec{r}} f(\vec{r}) \qquad (\star)$$

$$f(\vec{r}) = \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}\cdot\vec{r}} f(\vec{k}) \qquad (\star^4)$$

- First note that by complex conjugation

$$(e^{i\vec{k}\cdot\vec{r}})^* = e^{-i\vec{k}\vec{r}} = \sum_n J_n(kr)(-i)^n e^{-in\Delta\phi} \qquad (\star\star)$$

Here $\Delta\phi = \phi_k - \phi_r$

- Substituting (★★) into (★), expanding $f(\vec{r})$

$$f(\vec{r}) = \sum_m \frac{1}{2\pi} f_m(r)\, e^{im\phi_r}$$

- Using the relation

$$\int_0^{2\pi} \frac{d\phi_r}{2\pi}\, e^{+in\phi_r}\, e^{im\phi_r} = \delta_{m,-n}$$

- Which collapses the m-sum and sets $f_m(r) \to f_{-n}(r)$, yields finally:

$$f(\vec{k}) = \sum_n \int_0^\infty r\,dr\, J_n(kr)\,(-i)^n\, e^{-in\phi_k}\, f_{-n}(r)$$

Changing $n = -\ell$ using

$$J_{-\ell}(kr)(-i)^{-\ell} = (-1)^\ell J_\ell(kr)\, i^\ell$$

$$= (-i)^\ell J_\ell(kr)$$

Yields

$$f(\vec{k}) = \sum_\ell \int r\,dr\, J_\ell(kr)(-i)^\ell e^{i\ell\phi_k}\, f_\ell(r)$$

• Yielding, upon comparison ⓦ the fourier series:

$$f(\vec{k}) = \sum_{\ell} \frac{1}{2\pi} f_\ell(k) \, e^{i\ell\phi_k} ,$$

The result:

$$f_\ell(k) = 2\pi \int_0^\infty r\,dr \, J_\ell(kr) \, (-i)^\ell \, f_\ell(r) \qquad (\star\star\star)$$

To prove the second result we only need to realize that $(\star^4)$ is essentially the same up to $(2\pi)^2$ as $(\star)$. Exchanging k and r and taking the conjugate yields (after dividing by $(2\pi)^2$)

$$f_\ell(r) = \frac{2\pi}{(2\pi)^2} \int_0^\infty k\,dk \, J_\ell(kr) \, (+i)^\ell \, f_\ell(k)$$

$\boxed{\text{d)}}$ We have

$$\int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} = \frac{1}{r} \delta(r-r') \, \delta(\phi_r - \phi_r') \qquad (54)$$

• Then write the exponents as

$$e^{i\vec{k}\cdot\vec{x}} = \sum_n J_n(kr) \, i^n \, e^{in(\phi_k - \phi_r)}$$

$$e^{-i\vec{k}\cdot\vec{y}} = \sum_m J_m(kr') \, (-i)^m \, e^{-im(\phi_k - \phi_r')}$$

And integrate over ϕ_k using

$$\int \frac{d\phi_k}{2\pi} e^{in\phi_k} e^{-im\phi_k} = \delta_{nm}$$

- The δ_{nm} collapses the m-sum yielding

$$\int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} = \sum_n \int_0^\infty \frac{k\,dk}{2\pi} J_n(kr) J_n(kr') e^{-in(\phi_r-\phi_r')}$$

The r.h.s of Eq (54) is

compare these two

$$\sum_n \frac{1}{r'}\delta(r-r') e^{-in(\phi_r-\phi_{r'})}$$

- Comparison of the fourier series yields:

$$\int_0^\infty \frac{k\,dk}{2\pi} J_n(kr) J_n(kr') = \frac{1}{r}\delta(r-r')$$

- Now the consistency is straightforward:

$$f_n(r) = \int_0^\infty \frac{k\,dk}{2\pi} J_n(kr)\, i^\ell \underbrace{\left[2\pi \int_0^\infty r'dr' J_n(kr')(-i)^\ell f_n(r')\right]}_{\equiv f_n(k)}$$

$$= \int_0^\infty r'dr' \frac{1}{r}\delta(r-r') f_n(r')$$

$$f_n(r) = f_n(r) \checkmark$$

Problem 5

a) We expand

$$S(t) \equiv A\, e^{-i(\omega_c t + \varepsilon \cos(\omega_0 t))} \qquad (5.1)$$

In a Fourier series

$$S(t) = \sum_n \frac{1}{2T_0} S_n\, e^{-i\omega_n t} \qquad (5.2)$$

where $\omega_n = 2\pi n/T_0$. Here $\omega_c = 2\pi n_c/T_0$.

Then

$$S(t) = A\, e^{-i\omega_c t}\, e^{-i\varepsilon \cos(\omega_0 t)}$$

Expand

$$e^{-i\varepsilon \cos(\omega_0 t)} = \sum_n J_n(\varepsilon)\, (-i)^n\, e^{-in\omega_0 t}$$

So since $\omega_c = n_c \omega_0$ we find

$$S(t) = \sum_n A\, J_n(\varepsilon)\, (-i)^n\, e^{-i(n+n_c)\omega_0 t}$$

Let $l = n + n_c \qquad n = l - n_c$

Then

$$S(t) = \sum_{\ell} A \, J_{\ell - n_c}(\varepsilon) \, (-i)^{\ell - n_c} \, e^{-i \ell \omega_0 t} \tag{5.3}$$

So comparing (5.3) with (5.2) yields

$$\frac{S_\ell}{T_0 A} = J_{\ell - n_c}(\varepsilon) \, (-i)^{\ell - n_c} \tag{5.4}$$

i) For $\varepsilon \ll 1$ $\quad S(t) = A \, e^{-i \omega_c t}$

Then

$$\frac{S_\ell}{A T_0} = \delta_{\ell n_c} = \int_0^T \frac{S(t) \, e^{i \omega_0 \ell t}}{A T_0}$$

Then all the power is in the carrier frequency

ii) For $\varepsilon = 40$ the effective frequency ranges from $\omega_0 (820 - 40)$ to $(820 + 40)\omega_0$.

And the band width is $80 \omega_0$

iii) Recall that

$$\int_0^T dt\, |S(t)|^2 = \frac{1}{T_0} \sum_n |S_n|^2 \qquad (5.5)$$

- To prove this just substitute Eq (5.2) into the LHS of Eq. (5.5). For the current case this implies

$$\int_0^T d\tau\, |S(\tau)|^2 = A^2 T = \frac{1}{T} \sum_\ell J^2_{\ell-n_c}(\varepsilon)\, T^2 A^2$$

Or

$$\sum_\ell J^2_{\ell-n_c}(\varepsilon) = 1$$

- This is confirmed by the mathematica. In principle we should sum $\ell = -\infty \ldots +\infty$. But to good numerical we only need to sum where the S_n's have significant strength

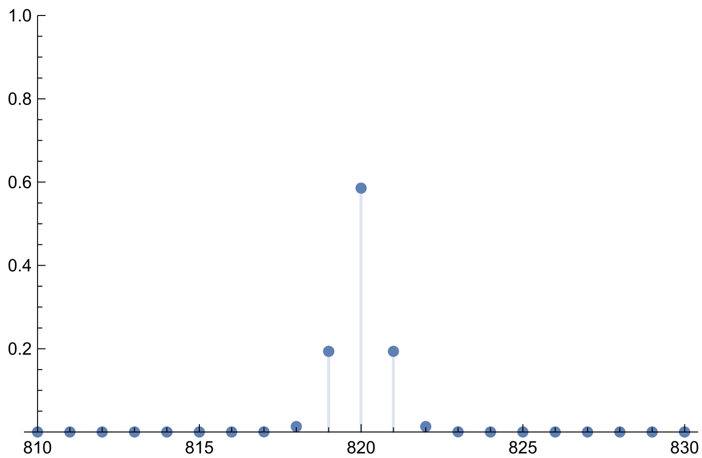
Homework 3 *Mathematica* Notebook

This is a discrete plot of the bessel sum for epsilon unity.

In[3]:=

```
DiscretePlot[BesselJ[n - 820, 1]^2, {n, 810, 830}, PlotRange → { 0, 1.} ]
```

Out[3]=

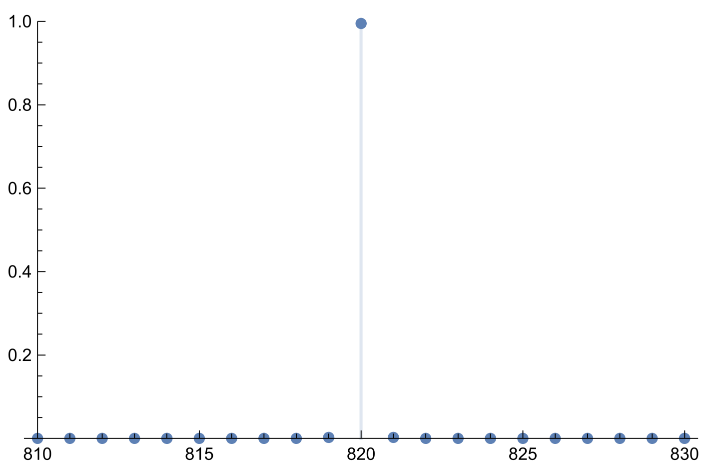


Now for epsilon = 0.1 we find that the power spectrum is very nearly unity for one fourier mode.

In[4]:=

```
DiscretePlot[BesselJ[n - 820, 0.1]^2, {n, 810, 830}, PlotRange → { 0, 1.} ]
```

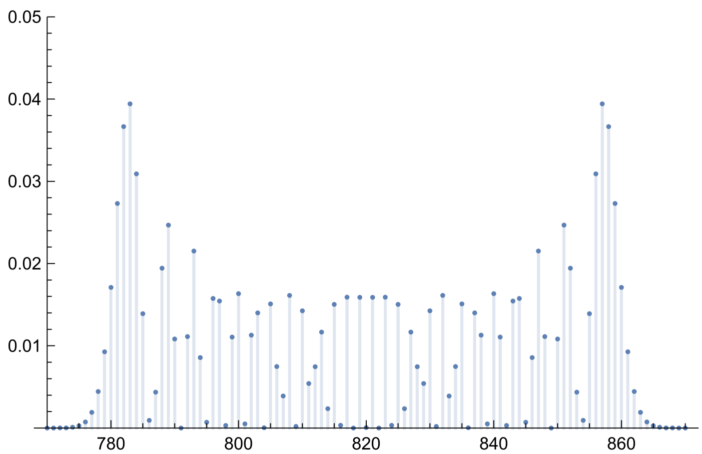
Out[4]=



Now for epsilon = 40 we find that the power spectrum is broader:

In[8]:=
```
DiscretePlot[BesselJ[n - 820, 40]^2, {n, 770, 870}, PlotRange → { 0, 0.05} ]
```

Out[8]=



The final part looks at the sum

In[10]:=
```
Sum[ BesselJ[n - 820, 40.]^2, {n, 700, 900 } ]
```

Out[10]= 1.