11.2.1 Arfken Cauchy Riemann

a) $f(z) = u + iV$

Recall

$$\frac{df}{dz} = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)(u + iv)$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial V}{\partial y}\right) + \frac{i}{2}\left(\frac{-\partial u}{\partial y} + \frac{\partial V}{\partial x}\right)$$

Now by Cauchy Riemann $\partial u/\partial x = \partial V/\partial y \quad -\partial u/\partial y = \partial V/\partial x$
and thus

$$\frac{df}{dz} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = V_x - i V_y$$

b) Take $\nabla \cdot V = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \nabla^2 u$

Now $\nabla^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

So $4\frac{\partial^2}{\partial z\,\partial \bar{z}}(u + iv) = 0$ since $\frac{\partial f}{\partial \bar{z}} = 0$

So $\nabla^2 u = \nabla^2 v = 0$

Now

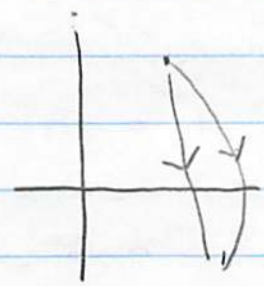
c $\quad \nabla \times V = \frac{\partial}{\partial x} V_y - \frac{\partial}{\partial y} V_x$

$$= \frac{\partial}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\frac{\partial u}{\partial x} = 0$$

Integrals

11.3.3, Consider a straight line path first $t \in [0,1]$

$$z(t) = z_1 + \Delta z\, t \quad \text{with} \quad z_2 - z_1 = \Delta z$$



Then for straight line:

$$\int_{z_1}^{z_2} dz\, z^n = \int_0^1 \Delta z\, dt\, (z_1 + \Delta z\, t)^n$$

$$= \frac{\cancel{\Delta z}\,(z_1 + \Delta z\, t)^{n+1}}{\cancel{\Delta z}\;\; n+1}\Bigg|_{t=0}^{1}$$

$$= \frac{z_2^{n+1}}{n+1} - \frac{z_1^{n+1}}{n+1}$$

we used that

$$z_1 + \Delta z t\Big|_{t=1} = z_2 \quad \text{and} \quad z_1 + \Delta z\, t\Big|_{t=0} = z_1$$

- Similarly for a circlular arc

$$z = r e^{i\theta} \qquad dz = r e^{i\theta} i \, d\theta$$

Then

$$\int_{z_1}^{z_2} dz \; z^n = \int_{\theta_1}^{\theta_2} r e^{i\theta} i d\theta \quad r^n e^{in\theta}$$

$$= r^{n+1} e^{i(n+1)\theta} \frac{i}{i(n+1)} \Bigg|_{\theta_1}^{\theta_2}$$

$$= \frac{z_2^{n+1}}{n+1} - \frac{z_1^{n+1}}{n+1}$$ ← Same as for straight path

where $z_2 = r e^{i\theta_2}$ and $z_1 = r e^{i\theta_1}$, Thus for either path:

$$\int_{z_1}^{z_2} dz \left(4z^2 - 3iz\right) = \frac{4}{3} z^3 - \frac{3}{2} i z^2 \Bigg|_{3+4i}^{4-3i}$$

$$= \frac{76}{3} - \frac{707}{3} i$$

← mathematica

To do the last and evaluate the numerical value at the two limits I used mathematica

b) • Then in the previous example

$$\frac{\partial f}{\partial \bar{z}} = 0 \qquad \text{where} \qquad f = 4z^2 - 3iz$$

The fact that $\partial f/\partial \bar{z} = 0$ show that the function is holomorphic

• While for the current case

$$f = \left(\frac{(z+\bar{z})}{2}\right)^2 + -i\left(\frac{(z-\bar{z})}{2i}\right)^2$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{z+\bar{z}}{2} - i\left(\frac{z-\bar{z}}{2i}\right) - \frac{1}{i}$$

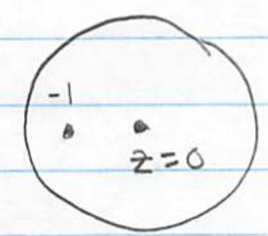
$$= x + y \neq 0$$ Now its not holomorphic

3 Arfken 11.3.7 - Integrals Around Loops

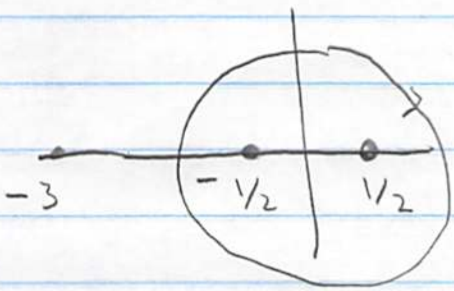
a) •

$$\oint_R \frac{dz}{z(z+1)} = 2\pi i \left[\operatorname*{Res}_{z=0} f + \operatorname*{Res}_{z=-1} f \right]$$

$$= 2\pi i \left[1 + -1 \right] = 0$$



b) • The poles are at $z=-3$, $z=-\frac{1}{2}$ and $z=+\frac{1}{2}$



$$f(z) = \frac{1}{8} \frac{1}{(z-1/2)(z+1/2)^2 (z+3)}$$

• The expansion of $f(z)$ near the poles $1/2$ and $-1/2$

① $$f(z) \simeq \frac{1}{8} \frac{1}{(z-1/2)} \frac{1}{(1/2+1/2)^2 (1/2+3)} + \ldots \qquad z \simeq 1/2$$

$$f(z) = \frac{1}{(z-1/2)} \cdot \frac{1}{28} + \ldots \qquad z \simeq 1/2$$

② $$f(z) = \frac{C}{(z+1/2)^2} + \frac{-3/100}{(z+1/2)} + \ldots \qquad z = -1/2$$

Used Mathematica here to determine this series:

to find the residue one just expands the function at the specified point. Mathematica is very good at this, e.g. `Series[1/(x - 1/2)], x, 0, 4]`

Then

clockwise path

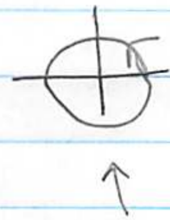
$$I = -2\pi i \left[\operatorname*{Res}_{z=1/2} f + \operatorname*{Res}_{z=-1/2} f \right] = -2\pi i \left[\frac{1}{28} - \frac{3}{100} \right]$$

$$= -2\pi i / 175$$

Trigonometric Integrals

a)

- $$I = \int_0^{2\pi} \frac{d\theta}{(1 + a\cos\theta)}$$



unit circle

$$= \int \frac{dz}{iz} \frac{1}{(1 + a(z + 1/z)/2)} \quad \leftarrow \cos\theta = z + \frac{1}{z}$$

$$= \frac{1}{i} \int dz \frac{1}{\frac{a}{2}z^2 + z + \frac{a}{2}}$$

- Now we look for the roots of the denominator

$$az^2 + 2z + a = 0$$

Which has roots:

$$z_\pm = \frac{-1 \pm \sqrt{1 - a^2}}{a}$$, but only the plus root lies within the circle:

$$f(z) = \frac{2}{az^2 + 2z + a} = \frac{2}{a(z - z_+)(z - z_-)}$$

$$\operatorname*{Res}_{z = z_+} f = \frac{2}{a(z_+ - z_-)} = \frac{2}{2\cancel{a}} \frac{\cancel{a}}{(1 - a^2)^{1/2}}$$

So

$$I = \frac{1}{i} \frac{2\pi i}{(1-a^2)^{1/2}} = \frac{2\pi}{(1-a^2)^{1/2}}$$

This clearly satisfies the minimal limit, i.e. that for $a = 0$ $I = 2\pi$

b) Now consider

$$I = \int_0^{2\pi} d\theta \, (1 - 2\cos\theta \, t + t^2)^{-1}$$

- with the substitution $\cos\theta = (z + 1/z)/2$ and $d\theta = dz/iz$ we have

$$I = \int_{\bigcirc} \frac{dz}{iz} \frac{1}{(1 - t(z + 1/z) + t^2)}$$

- Thus we are led to looking for roots of :

$$-t z^2 + (1 + t^2) z - t = 0$$

which has rootz $z_> = \frac{1}{t}$ and $t \equiv z_<$ nealty.

- Then

$$I = \int_{\bigcirc} \frac{dz}{i} \frac{1}{-t\,[z - z_<][z - z_>]}$$

• Then only $z_< \equiv t$ is in the circle

$$I = \frac{2\pi i}{-it} \operatorname{Res}_{z_<} \frac{1}{(z-z_<)(z-z_>)}$$

$$I = \frac{2\pi}{-t\,(z_< - z_>)} = \frac{2\pi}{-t\,(t - 1/t)} = \frac{2\pi}{(1-t^2)}$$

• c) The integral

$$I = \int_0^{2\pi} \frac{d\theta}{(1 + a\cos\theta)}$$ diverges for $a \geq 1$

when the denominator can vanish. Thus when viewed as a complex function of a we expect a singularity for $a \to 1$. This is what is seen in the answer:

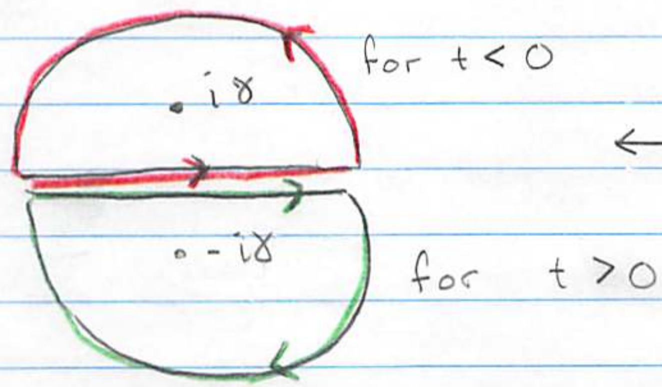
$$I(a) = \frac{2\pi}{(1-a^2)^{1/2}}$$

Problem 5 - Fourier Integrals

$$I(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{2\gamma}{\omega^2+\gamma^2}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{2\gamma}{(\omega+i\gamma)(\omega-i\gamma)}$$

- For $t<0$ $e^{-i\omega t} \xrightarrow[\omega\to+i\infty]{} 0$, while for $t>0$ we have $e^{-i\omega t} \xrightarrow[\omega\to-\infty]{} 0$. Thus we have to close the contour above for $t<0$, and close below for $t>0$. ← See picture.



- For $t<0$ we find, $f(z) \equiv (2\gamma/z^2+\gamma^2)\, e^{-i\omega t}$:

$$I(t) = \frac{1}{2\pi} 2\pi i \operatorname*{Res}_{z=i\gamma} f(z) = e^{\gamma t}$$

Near $i\gamma$ we analyzed $f(z)$ as follows:

this thing is the residue by definition

$$f = \frac{2\gamma}{(z+i\gamma)} \frac{1}{(z-i\gamma)} e^{-i\omega t} \simeq \left(\frac{2\gamma}{2i\gamma} e^{-i(i\gamma t)}\right) \frac{1}{z-i\gamma}$$

- Similarly for $t>0$ we have

$$I(t) = \frac{1}{2\pi}(-2\pi i)\, \underset{-i\gamma}{\mathrm{Res}}\, f(z) = e^{\gamma t}$$

"wrong"/clockwise circulation of pole

Near $-i\gamma$ we extracted the Residue of $f(z)$ as:

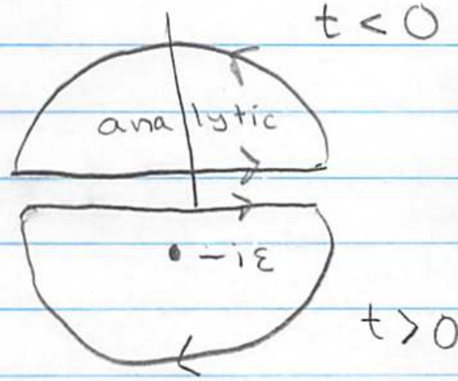
$$f(z) = \frac{2\gamma}{(z+i\gamma)} \frac{1}{(z-i\gamma)} e^{-i\omega t} \simeq \underbrace{\frac{2\gamma}{-2i\gamma} e^{-i(-i\gamma)t}}_{\text{this is the residue}} \frac{1}{(z+i\gamma)}$$

- Thus

$$I(t) = \begin{cases} e^{\gamma t} & t<0 \\ e^{-\gamma t} & t>0 \end{cases}$$

b) $$I_\varepsilon(t) \equiv -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{1}{\omega+i\varepsilon} e^{-i\omega t}$$

- The arguement is very similar to the previous item. For $t>0$ we close below (and will pick up a pole) while for $t<0$ we close above (and the function is analytic there)



for t > 0

- Thus ^ wrong way around pole

$$I_\varepsilon(t) = -2\pi i \operatorname*{Res}_{z=-i\varepsilon} \frac{-1}{2\pi i} \frac{1}{(\omega+i\varepsilon)} e^{-i\omega t}$$

$$= e^{-i(-i\varepsilon)t} = e^{-\varepsilon t}$$

no singularity if $\mathrm{Im}\,\omega \geq 0$

- While for $t<0$ our function $\dfrac{1}{\omega+i\varepsilon}$ is analytic in the upper half plane. For an analytic function.

$$\oint f(z) = 0$$

- Summarizing

$$I_\varepsilon(t) = \Theta(t)\, e^{-\varepsilon t}$$ ← for $t=0$ the integral is ambiguous and gives $\frac{1}{2}$

c) Now

$$\frac{d}{dt}\Theta(t) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{i}{\omega + i\varepsilon} \overbrace{\frac{d}{dt} e^{-i\omega t}}^{-ie^{-i\omega t}\omega}$$

$$\delta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} = \delta(t)$$

a) $\boxed{\text{i)}}$ $\frac{1}{(z^2-4)^{1/2}}$ here $\underline{R = 2}$, The nearest singularities are at $z = \pm 2$.

$\boxed{\text{ii)}}$ $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$

$= z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$

Then R.O.C is:

$(R)^{-1} = \limsup_{n \to \infty} \left(\frac{1}{n!}\right)^{1/n} = \lim_{n \to \infty} \frac{1}{(n^n e^{-n})^{1/n}} = \frac{1}{n e^{-1}}$

$R^{-1} = 0$ or $\underline{R = \infty}$.

$\boxed{\text{iii)}}$ Then $\log(z+4)$ has its nearest singularity at $z = -4$. R.O.C is therefore 4.

b) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,n!} \left(\frac{z}{2}\right)^{2n}$

So comparison with the canonical $\sum_k^{\infty} a_k z^k$ gives

$|a_k| = \frac{1}{(k!\,k!)^{1/2}\, 2^k} = \frac{1}{k!\, 2^k}$

Then using the formula,

$$\lim_{k \to \infty} |a_k|^{1/k} = \frac{1}{(k^k e^{-k} \cdot 2k)^{1/k}} = \frac{1}{k e^{-1} \cdot 2}$$

Thus $|a_k|^{1/k} \longrightarrow 0$ as $k \to \infty$

Thus, $R = \dfrac{1}{\lim_{k \to \infty} |a_k|^{1/k}} = \infty$