Problem 1. Integrals with Principal Values and Branch Singularities

Do the following integrals:

(a) By considering $(1 - e^{2ix})/x^2$ show that

$$\int_0^\infty dx \left(\frac{\sin x}{x}\right)^2 = \frac{\pi}{2} \tag{1}$$

(b) Next we consider two related itengrals

(i) Evaluate

$$\int_0^\infty dx\, \frac{x^{a-1}}{1+x} = \frac{\pi}{\sin(\pi a)} \tag{2}$$

(ii) Evaluate

$$\int_0^\infty x^{a-1} \frac{\mathcal{P}}{1-x^2} = \frac{\pi(1+\cos(\pi\alpha))}{2\sin(\pi\alpha)} \tag{3}$$

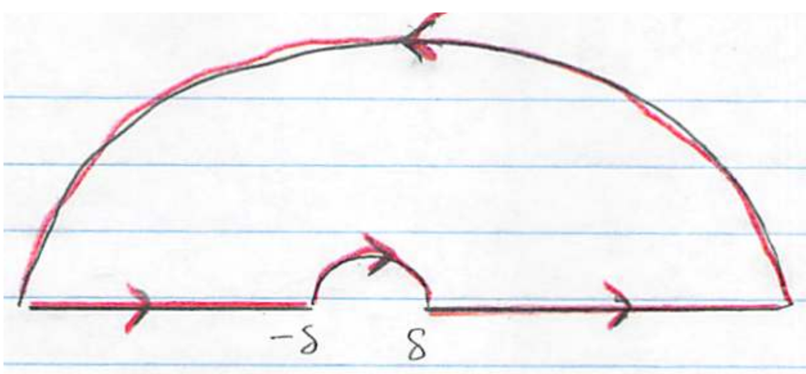
where P denotes the principal value distribution. Why does Eq. (2) have a pole at $a = 1$ while Eq. (3) does not? One way to procede is to partial fraction

$$\frac{1}{1-x^2} = \frac{1}{2}\frac{1}{1-x} + \frac{1}{2}\frac{1}{1+x} \tag{4}$$

(c) Evalute

$$\int_0^\infty dx \frac{(\log x)^2}{1+x^2} = \frac{\pi^3}{8} \tag{5}$$

Hint use the contour shown below and for simplicity take the cut of the logarithm from $(0 - i\infty, 0)$



Problem 2. Analytic continuation of $\text{atanh}(z)$ and friends

The problem has many (often easy) parts. Write as little as possible. Just enough to show that you know what is going on.

(a) First show that for a real numbers[1] y

$$\sin(iy) = i\sinh(y) \tag{10}$$
$$\cos(iy) = \cosh(y) \tag{11}$$
$$\tan(iy) = i\tanh(y) \tag{12}$$

and determine the Taylor series of

$$\tanh(y) = y + \ldots \tag{13}$$

to third order in small y inclusive.

(b) From its definition

$$\tanh(y) \equiv \frac{e^y - e^{-y}}{e^y + e^{-y}} \tag{14}$$

show by elementary algebra that

$$y = \text{atanh}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) . \tag{15}$$

Other formulas you should know how to derive (see part (g)) are

$$\text{asinh}(x) = \log\left(x + \sqrt{x^2+1}\right) \tag{16}$$
$$\text{acosh}(x) = \log\left(x + \sqrt{x^2-1}\right) \tag{17}$$

These are useful below.

(c) Determine the taylor series of

$$\text{atanh}(x) = x + \ldots \tag{18}$$

near the origin to third order in x.

[1] These hold for complex numbers as well, and lead to many familiar identities, e.g.

$$\cosh^2(z) - \sinh^2(z) = 1 \tag{6}$$
$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) \tag{7}$$
$$\cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y) \tag{8}$$
$$\tag{9}$$

etc

(d) By composing the taylor series in Eq. (13) and Eq. (18), verify that through third order inclusive we have

$$\tanh(\text{atanh}(x)) = x + O(x^4) \tag{19}$$

(e) Then use elementary means to show for $|x| < 1$ on the real axis that

$$\int_0^x \frac{dx'}{1 - x'^2} = \text{atanh}(x) \tag{20}$$

Extending these results to the complex plane, $\text{atanh}(z, \gamma)$ for a point z in the complex plane and path γ is *defined* as

$$\text{atanh}(z, \gamma) = \int_\gamma \frac{dz'}{1 - z'^2} \tag{21}$$

where the path $\gamma(z')$ connects the origin, $z' = 0$, to the point $z' = z$. Depending on how many times the path encircles the branch points at $1, -1$, you will get different answers for this function. $\text{atanh}(z, \gamma)$ thus depends on the path only through the path's topology.

Different choices for the cut lines conventionally dictate the *canonical* value of this function at a point z by specifying the topology of the allowed path. Specifically the canonical value (for a given choice of cut) is found by requiring that the path should not cross the cut line. This limits a path's topology.

Naively integrating Eq. (21) yields

$$\text{atanh}(z, \gamma) = \frac{1}{2} \log\left(\frac{1 + z}{1 - z}\right) \tag{22}$$

The result is ambiguous until the paths are defined which give definite meaning to the logarithims as

$$\frac{1}{2} \log\left(\frac{1 + z}{1 - z}\right) \equiv \frac{1}{2} \int_\gamma \frac{dz'}{1 + z'} + \frac{1}{2} \int_\gamma \frac{dz'}{1 - z'} \tag{23}$$

$$= \int_\gamma \frac{dz'}{1 - z'^2} \tag{24}$$

Here $\text{atanh}(z, \gamma)$ is defined so that at[2] $0 + i\epsilon$ (our starting point) $\text{atanh}(z, \gamma)$ agrees with the power series given by Eq. (18). Compare three different choices for the branch cuts:

A. The branch cuts are chosen to be on the real axis $(-\infty, -1)$ and $(1, \infty)$. Lets call this choice A. (This is what is used in `Mathematica`.)

B. Another equally valid choice of branch cuts are the lines in the imaginary directions, $(-1 - i\infty, -1)$ and $(1 - i\infty, 1)$. Lets call this choice B.

C. (**Optional**) **Analyze this one for yourself but do not include it in your homework – it will not be graded.** Still another choice are the lines along the real axis $(-\infty, -1)$ and $(-\infty, 1)$. Lets call this choice C.

[2] Here and below ϵ notates a small positive real number

(f) Draw the three choices of cuts. For each of our three cut choices, determine the canonical value of atanh(z) at the following points, i.e. fill in the table. Some entries are filled to provide some answers but you should explain these values.

Point	Cut A	Cut B	Cut C (Optional)
$0 + i\epsilon$	0 (definition)	0 (definition)	0 (definition)
$0 - i\epsilon$			$i\pi$ (explain me)
$3 + i\epsilon$	$\frac{1}{2}\log(2) + i\frac{\pi}{2}$ (explain me)		
$3 - i\epsilon$			
$1 - 2i + \epsilon$			
$1 - 2i - \epsilon$			
$-3 + i\epsilon$			
$-3 - i\epsilon$			

(You can check your results for column A by comparing with `Mathematica`)

(g) Along the lines of part (a) show by algebraic means that

$$\text{asin}(x) = \frac{1}{i}\log(ix + \sqrt{1 - x^2}) \tag{25}$$

You can check with `Mathematica` that the two have the same series,

```
Series[ArcSin[x], {x, 0, 11}]

x + x^3/6 + 3 x^5/40 + 5 x^7/112 + 35 x^9/1152 + 63 x^11/2816 + O[x]^12

Series[1 / I Log[I x + Sqrt[1 - x^2]], {x, 0, 11}]

x + x^3/6 + 3 x^5/40 + 5 x^7/112 + 35 x^9/1152 + 63 x^11/2816 + O[x]^12
```

(h) Show that for real x for $|x| < 1$

$$\text{asin}(x) = \int_0^x \frac{dx'}{\sqrt{1 - x'^2}} \tag{26}$$

and

$$\int_0^1 \frac{dx'}{\sqrt{1 - x'^2}} = \frac{\pi}{2} \tag{27}$$

For complex value z *define*

$$\text{asin}(z, \gamma) \equiv \int_\gamma \frac{dz'}{\sqrt{1 - z'^2}} \tag{28}$$

where we take a path $\gamma(z')$ to connect the origin $z' = 0$ to point $z' = z$.

(i) Show that for real a with $a > 1$ we have the real integrals

$$\int_1^a \frac{dx}{\sqrt{x^2-1}} = \log(a + \sqrt{a^2-1}) \tag{29}$$

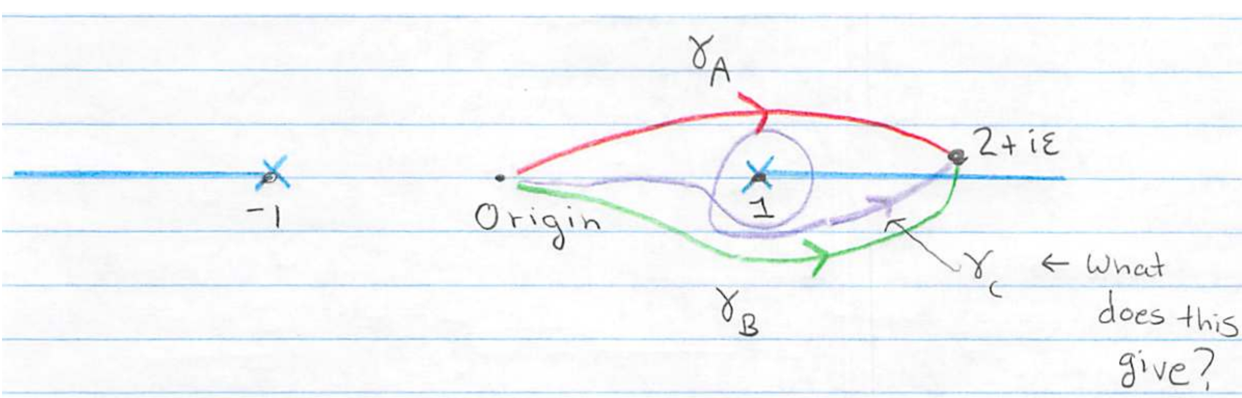
Hint: look at the part (a) especially the footnote. **Remark:** Often when dealing with the forms $\sqrt{x^2-1}$ and $\sqrt{x^2+1}$ or $\frac{1}{1+x^2}$ and unbounded intervals, hyperbolic trig substitutions are a better choice than trig substitions.

(j) First define a canonical value to $\operatorname{asin}(2+i\epsilon, \gamma)$ by placing a branch cut from $(-\infty, -1)$ and $(1, \infty)$ (You can check your result with `Mathematica` which uses the same choices):

$$\text{Ans} : \frac{\pi}{2} + i\log(2+\sqrt{3}) \tag{30}$$

Here we have taken the canonical path γ_A from the origin to $2 + i\epsilon$.

(k) What are $a$ll possible values of $\operatorname{asin}(2+i\epsilon, \gamma)$? For each possible value list the winding numbers (relative to the canonical path) around each branch point. For instance, you definitely should consider path γ_B. Relative to path γ_A (the canonical path for this choice of branch cuts) path B has a winding number $+1$ around the branch point $z = 1$, since the combined path[3] $\gamma_B \oplus -\gamma_A$ encircles the the branch point at $z = 1$ once in a counter-clockwise fashion. Similarly, consider the path γ_C (which relative to γ_A has winding number $W_{z=1} = +2$) and discuss its value. Draw other paths which encircle -1 and 1 in arbtitrary ways and determine their values.



(l) We have given two examples of how to define $\log(f(z), \gamma)$. Specifically we always define as

$$\log(f(z), \gamma) \equiv \int_\gamma dz \frac{f'(z)}{f(z)} \tag{31}$$

Thus

$$\frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \equiv \int_\gamma \frac{dz'}{1-z'^2}\,. \tag{32}$$

[3] $-\gamma_A$ notates a path in the opposite direction of A

Similarly

$$\operatorname{asin}(z) = \frac{1}{i}\log(iz + \sqrt{1-z^2}) \equiv \int_\gamma \frac{dz'}{\sqrt{1-z'^2}} \tag{33}$$

How would you define

$$\operatorname{acosh}(z) \tag{34}$$

for an arbitrary point in the plane as an integral.

Problem 3. Kramers Kronig

In class we showed that the Kramers-Kronig relations read

$$\mathrm{Re}G_R(\omega) = \mathcal{P}\int \frac{d\omega'}{\pi}\frac{\mathrm{Im}G_R(\omega)}{\omega'-\omega} \tag{35}$$

$$\mathrm{Im}G_R(\omega) = -\mathcal{P}\int \frac{d\omega'}{\pi}\frac{\mathrm{Re}G_R(\omega)}{\omega'-\omega} \tag{36}$$

where $G_R(\omega)$ is a Fourier transform of a causal function

$$G_R(\omega) = \int_0^\infty d\tau\, e^{+i\omega\tau} G_R(\tau) \tag{37}$$

(a) Show that if $G_R(\tau)$ is a real function Eq. (35) can be written as

$$\mathrm{Re}G_R(\omega) = \frac{2}{\pi}\mathcal{P}\int_0^\infty d\omega' \frac{\omega' \mathrm{Im}G_R(\omega')}{\omega'^2-\omega^2} \tag{38}$$

$$\mathrm{Im}G_R(\omega) = -\frac{2\omega}{\pi}\mathcal{P}\int_0^\infty d\omega' \frac{\mathrm{Re}\mathrm{G}_\mathrm{R}(\omega')}{\omega'^2-\omega^2} \tag{39}$$

Hint if $G_R(\tau)$ is real, then its Fourier transform has certain properties.

Take a response function of a damped harmonic oscillator

$$G_R(\omega) = \frac{-1}{2}\frac{1}{\omega-\omega_o+i\Gamma} + \frac{1}{2}\frac{1}{\omega+\omega_o+i\Gamma} \tag{40}$$

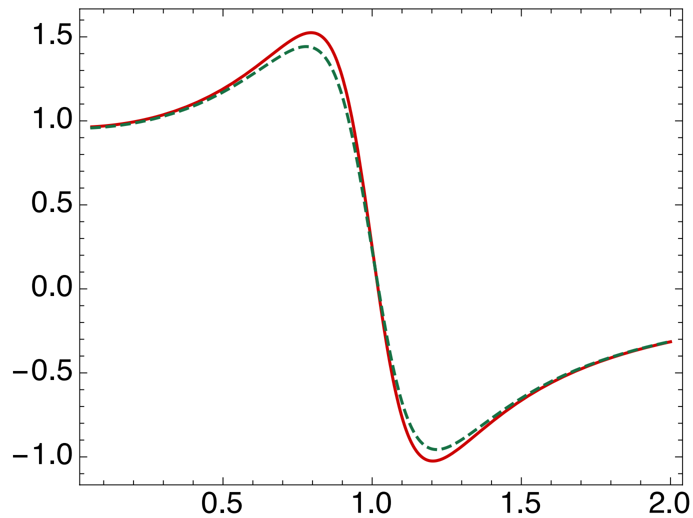
Next week we will show by contour integration that in coordinate space this corresponds to the following real causal function derived in class

$$G_R(\tau) = \theta(\tau)e^{-\Gamma\tau}\sin(\omega_o\tau) \tag{41}$$

with $\tau = t - t_o$. You may wish to do this excercise now, but save the solution for next week.

(b) Check the Kramers-Kronig numerically for the response function of a damped harmonic oscillator (Eq. (40)) using `Mathematica` with $\omega_0 = 1$ and $\Gamma = 0.2$. Specifically, integrate the imaginary part along the real axis using Eq. (38) and show numerically that it equals the analytical real part. (Needless to say *do not* use Mathematica's built in principal value capabilities). To do the integral you will need to choose some way to regulate the principal value distribution. A simple way in this case is just to integrate up to $\omega - 10\epsilon$ and then integrate from $\omega + 10\epsilon$ for some small value (say $\epsilon = 0.001$) which you can adjust.

Make a plot of your numerical real part from $0.1 \ldots 2$ and the analytical real part for this response function in the same range. You should find with $\epsilon = 0.002$



I attatch a sample mathematica notebook that does the random integral

$$I(a) = \mathcal{P} \int_{-\pi/2}^{\pi/2} dx \, \frac{e^{ax}}{x} \tag{42}$$

so you can see the necessary `Mathematica` commands.

Help with mathematica:

For numerical work define functions as with ?NumericQ, or it wont work a lot of the time. This tells mathematica that we are expecting a number here

```
f[x_?NumericQ, a_?NumericQ] := Exp[a x] / x
```

```
In[99]:= { f[y, 1], f[3., 1.] }  (* f[x,a] will only operate if given a number *)
Out[99]= {f[y, 1], 20.0855}
```

Here I show how (very primatively) to do a numerical principal value integral

```
In[116]:= myintegral[a_?NumericQ, eps_?NumericQ] :=
  NIntegrate[f[x, a], {x, -Pi / 2 , -5 eps } ] + NIntegrate[f[x, a], {x, 5 eps, Pi / 2} ]
```

Here I show how the value of myintegral for different values of “a” and the smoothening parameter “epsilon”, eps=1.e-4 and eps=1.e-2.

```
In[115]:= Plot[ {myintegral[a, 10.^(-4)], myintegral[a, 10.^(-2)] }, {a, 1, 2} ]
```

Out[115]=