Problem 1

	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

$\boxed{(b)}$ · Then the group is commuting so

$$a = g a g^{-1} \quad \text{for all} \quad g \in G$$

So each element forms it's own class

· Since the group is commuting any matrix of a rep commutes with its partners

$$[D(g), D(g')] = 0$$

for all $g' \in G$ this says $D(g) = \lambda \mathbb{1}$ by schurs lemma. Thus $D(g)$ is reducable or is simply one dimensional.

$\boxed{c)}$

	e	a	b	ab
e	1	1	1	1
a	1	1	-1	-1
b	1	-1	1	-1
ab	1	-1	-1	1

- We can easily see row orthogonality and column orthogonality

e.g. comparing row (1) and row (2)

$$1\cdot 1 + 1\cdot 1 + 1\cdot(-1) + 1\cdot(-1) = 0$$

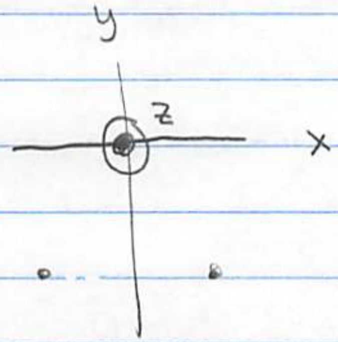
comparing row (1) and row (1)

$$1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 1\cdot(1) = 4 \leftarrow n_G$$

- Similarly comparing column 1 and column 2

$$1\cdot 1 + 1\cdot 1 + 1\cdot -1 + 1\cdot(-1) = 0$$

d) Then



$$(x_1, y_1, z_1, x_2 y_2 z_2) \xrightarrow[a]{} (-x_2, y_2, -z_2, -x_1, y_1, -z_2)$$

$$(x_1, y_1, z_1, x_2 y_2 z_2) \xrightarrow[b]{} (-x_2, y_2, z_2, -x_1, y_1, z_1)$$

$$(x_1 y_1 z_1 ; x_2 y_2 z_2) \xrightarrow[ab]{} (x_1 y_1, -z_1, x_2 y_2 -y_2)$$

So as matrices

$$O(e) = \mathbb{1}_{6\times 6}$$

$$O(a) = \left(\begin{array}{ccc|ccc} & & & -1 & & \\ & 0 & & & 1 & \\ & & & & & -1 \\ \hline -1 & & & & & \\ & 1 & & & 0 & \\ & & -1 & & & \end{array}\right)$$

$$O(b) = \left(\begin{array}{ccc|ccc} & & & -1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \\ \hline -1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \end{array}\right)$$

$$O_{ab} = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & 0 & \\ & & -1 & & & \\ \hline & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & -1 \end{array}\right)$$

So

$\chi(e) = 6$

$\chi(ab) = 2$

$\chi(a) = 0$

$\chi(b) = 0$

e) Then Character analysis gives

$$\chi(g) = \sum_{\mu} a_{(\mu)} \chi^{(\mu)}(g) \qquad a_{\mu} = \frac{1}{n_G} \sum_{g} \chi(g)(\chi^{(\mu)}(g))^*$$

$$a_{(1)} = \frac{1}{4}(6\cdot 1 + 2\cdot 1) = 2$$

$$a_{(2)} = \frac{1}{4}(6\cdot 1 + 2\cdot -1) = 1$$

$$a_{(3)} = \frac{1}{4}(6\cdot 1 + 2\cdot -1) = 1$$

$$a_{(4)} = \frac{1}{4}(6\cdot 1 + 2\cdot 1) = 2$$

So

$$O_{ab} \to \underline{O}_{ab} = 2D^{(1)} \oplus D^{(2)} \oplus D^{(3)} \oplus 2D^{(4)}$$

(f) Then

$$\vec{\phi}_1^{(1)} = \frac{1}{4}\sum_g \hat{e}^{(1)} \vec{\phi}_1 = \boxed{\frac{1}{2}(1,0,0,-1,0,0) = \vec{\phi}_1^{(1)}}$$

returns $(1,0,0,0,0,0)$

$$\vec{\phi}_1^{(1)} = \frac{1}{4}(\vec{\phi}_1 + O_a\vec{\phi}_1 + O_b\vec{\phi}_1 + O_{ab}\vec{\phi}_1)$$

each of these give $(0,0,0,-1,0,0)$

$$\vec{\phi}_1^{(2)} = \frac{1}{4}\sum_g \hat{e}^{(2)} \vec{\phi}_1 = 0$$

these cancel

$$= \frac{1}{4}(\vec{\phi}_1 + O_a\vec{\phi}_1 - O_b\vec{\phi}_1 - O_{ab}\vec{\phi}_1) = \boxed{\vec{0} = \vec{\phi}_1^{(2)}}$$

cancel

Similarly

$$\vec{\phi}_1^{(3)} = 0$$

$$\phi_1^{(4)} = (\vec{\Phi}_1 - O_a\vec{\phi}_1 - O_b\vec{\phi}_1 + O_{ab}\vec{\phi}_1)/4$$

$$\boxed{\phi_1^{(4)} = \frac{1}{2}(1,0,0,+1,0,0)}$$

One can check

$$\vec{\phi}_1 = \vec{\Phi}_1^{(1)} + \vec{\phi}_1^{(4)}$$

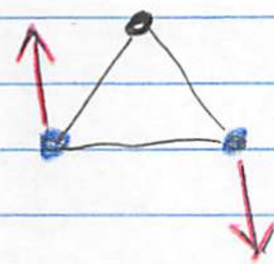
Picture:

Similarly

$$\left.\begin{array}{l} \boxed{\vec{\phi}_2^{(1)} = \frac{1}{2}(0,1,0,0,1,0)} \\ \vec{\phi}_2^{(2)} = \vec{0} \\ \vec{\phi}_2^{(3)} = \vec{0} \\ \boxed{\vec{\Phi}_2^{(4)} = \frac{1}{2}(0,1,0\,;\,0,-1,0)} \end{array}\right\} \quad \vec{\phi}_2' = \vec{\phi}_2^{(1)} + \vec{\phi}_2^{(4)}$$

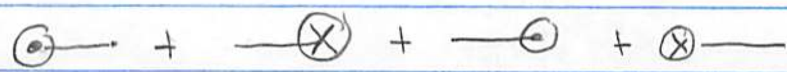
$\vec{\phi}_2^{(1)}$

$\vec{\phi}_2^{(4)}$



Finally

$$\vec{\Phi}_3^{(1)} = \frac{1}{4}\left(\vec{\Phi}_3 + O_a\vec{\Phi}_3 + O_b\vec{\Phi}_3 + O_{ab}\vec{\Phi}_3\right)$$

$= \frac{1}{4}\Big($  $\Big)$

$$\boxed{\vec{\Phi}_3^{(1)} = 0}$$

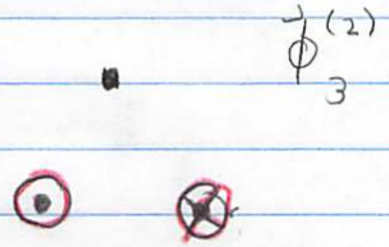
$$\vec{\Phi}_3^{(2)} = \frac{1}{4}\left(\vec{\Phi}_3 + O_a\vec{\Phi}_3 - O_b\vec{\Phi}_3 - O_{ab}\vec{\Phi}_3\right)$$

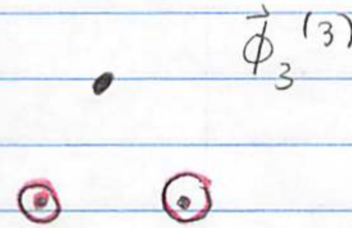
$$\boxed{\vec{\Phi}_3^{(2)} = \frac{1}{2}(0,0,1,0,0,-1)}$$

$$\Phi_3^{(3)} = \frac{1}{4}\left(\vec{\Phi}_3 - O_a\vec{\Phi}_3 + O_b\vec{\Phi}_3 - O_{ab}\vec{\Phi}_3\right)$$

$$\vec{\Phi}_3^{(3)} = \frac{1}{2}(0,0,1,0,0,1)$$

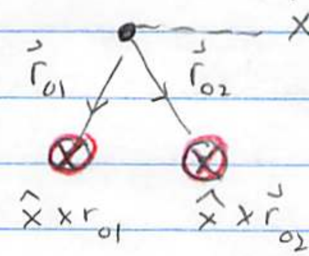
Finally $\boxed{\vec{\Phi}_3^{(4)} = \vec{0}}$





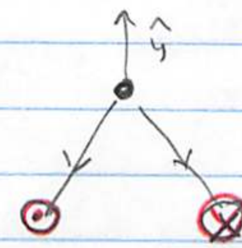
g) We have

i) $$\vec{\psi}_{0x} \propto (\hat{x} \times \vec{r}_{01} ; \hat{x} \times \vec{r}_{02})$$



$$\boxed{\vec{\psi}_{0x} \propto (0,0,-1 ; 0,0,-1)}$$

$$\vec{\psi}_{0y} \propto (\hat{y} \times \vec{r}_{01} ; \hat{y} \times \vec{r}_{02})$$



$$\boxed{\vec{\psi}_{0y} \propto (0,0,1 ; 0,0,-1)}$$

Finally

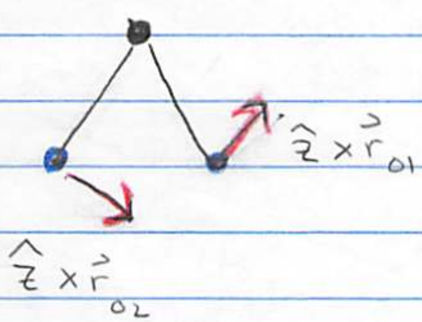
$$\vec{\psi}_{0z} = (\hat{z} \times \vec{r}_{01} ; \hat{z} \times \vec{r}_{02})$$

$$\vec{r}_{01} = -\sin\theta\, \hat{x} - \cos\theta\, \hat{y} \qquad \vec{r}_{02} = \sin\theta\, \hat{x} - \cos\theta\, \hat{y}$$

$$\hat{z} \times \vec{r}_{01} = -\sin\theta\, \hat{y} + \cos\theta\, \hat{x}$$

$$\hat{z} \times \vec{r}_{02} = \sin\theta\, \hat{y} + \cos\theta\, \hat{x}$$

Picture



we have inserted $1/2$ so $\vec{\psi}_{0z}^{T} \cdot \vec{\psi}_{0z} = 1$

$$\boxed{\vec{\psi}_{0z} = \frac{1}{\sqrt{2}} (\cos\theta, -\sin\theta, 0; \cos\theta, \sin\theta, 0)}$$

ii) Clearly

$$\boxed{\begin{aligned} \phi_3^{(2)} &\propto \vec{\psi}_{0y} \\ \phi_3^{(3)} &\propto -\vec{\psi}_{0x} \end{aligned}}$$

(h) $\boxed{\vec{\psi}_{0z} = \cos\theta \, \hat{\phi}_1^{(4)} - \sin\theta \, \hat{\phi}_2^{(4)}}$

Then the orthogonal vector is clearly

$$\psi_1^{(4)} = +\sin\theta \, \hat{\phi}_1^{(4)} + \cos\theta \, \hat{\phi}_2^{(4)}$$

We have normalized $\hat{\phi}_1^{(4)}$ and $\hat{\phi}_2^{(4)}$

$$\hat{\phi}_1^{(4)} = \frac{1}{\sqrt{2}} (1, 0, 0; 1, 0, 0)$$

To show explicitly that $\vec{\psi}_{0z}$ is a zero eigenmode we need to compute the "Hamiltonian" matrix;

$$U = \frac{1}{2} \frac{\partial U}{\partial q^a \partial q^b} q^a q^b = \frac{1}{2} H_{ab} q^a q^b$$

$$= \frac{1}{2} k (\vec{r}_{01} \cdot \vec{r}_1)^2 + \frac{1}{2} k (\vec{r}_{02} \cdot \vec{r}_2)^2 + \frac{1}{2} k (x_1 - x_2)^2$$

Since

$$\hat{r}_{01} = -\sin\theta\, \hat{x} - \cos\theta\, \hat{y} \qquad \hat{r}_{02} = \sin\theta\, \hat{x} - \cos\theta\, \hat{y}$$

We have

$$u = \frac{k}{2} \Big[(\sin^2\theta + 1) x_1^2 + \cos^2\theta\, y_1^2 + 2 \sin\theta \cos\theta\, x_1 y_1$$

$$+ (\sin^2\theta + 1) x_2^2 + \cos^2\theta\, y_2^2 - 2 \sin\theta \cos\theta\, x_2 y_2 - 2 x_1 x_2 \Big]$$

So the matrix $H \vec{\psi}_{0z}$ reads

$$H \vec{\psi}_{0z} = k \begin{pmatrix} s^2+1 & s \cdot c & 0 & -1 & 0 & 0 \\ sc & c^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & s^2+1 & -sc & 0 \\ 0 & 0 & 0 & -sc & c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ -s \\ 0 \\ c \\ s \\ 0 \end{pmatrix}$$

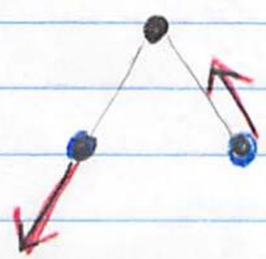
$$= \vec{0}$$ as claimed!

As described above, the orthogonal vector is

$$\psi_1^{(4)} = \sin\theta \; \hat{\phi}_1^{(4)} + \cos\theta \; \hat{\phi}_2^{(4)}$$

where

$$\hat{\psi}_1^{(4)} = \frac{1}{\sqrt{2}} (\sin\theta, \cos\theta, 0 ; \sin\theta, -\cos\theta, 0)$$



$\boxed{i)}$ The Oscillation frequency is

$$\omega_\lambda = \sqrt{\frac{k_\lambda}{m}} \quad \text{where} \quad k_\lambda = \vec{\psi}_1^{(4)} \cdot H \vec{\psi}_1^{(4)}$$

Straight forward algebra showes

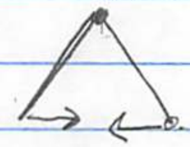
$$\vec{\psi}_1^{(4)} \cdot H \psi_1^{(4)} = k$$

So

$$\boxed{\omega_1^{(4)} = \sqrt{\frac{k}{m}}}$$

j) We now write the hamiltonian in the basis $\vec{\phi}_1^{(1)}$ and $\vec{\phi}_2^{(1)}$ which are displayed below

$\vec{y}_1 = \vec{\phi}_1^{(1)}$ $\qquad\qquad$ $\vec{\phi}_2^{(1)} \equiv \vec{y}_2$

- Then we have

$$H_{ab} = (\vec{y}_a : H \vec{y}_b) = \begin{pmatrix} 2+s^2 & sc \\ sc & c^2 \end{pmatrix}$$

$$= \frac{3}{2}\,\mathbb{1} + \left(\frac{3}{2} - \cos^2\theta\right)\sigma^z + \frac{1}{2}\sin 2\theta\,\sigma^x$$

$$= \frac{3}{2}\,\mathbb{1} + \left(1 - \frac{1}{2}\cos 2\theta\right)\sigma^z + \frac{1}{2}\sin 2\theta\,\sigma^x$$

- From which we read

$$\boxed{\omega_\pm^2 = \left(\frac{3}{2} \pm \sqrt{\frac{5}{4} - \cos(2\theta)}\right)\cdot\frac{k}{m}}$$

where $\frac{5}{4} - \cos 2\theta = \left(1 - \frac{1}{2}\cos 2\theta\right)^2 + \left(\frac{1}{2}\sin 2\theta\right)^2$

Problem 2. Inner product

For definiteness consider the canonical D_3 (or triangle) group that we discussed in class in two spatial dimensions. Take a inner product of two functions as simply

$$\langle f, h \rangle = \int d^2\boldsymbol{x}\, f^*(\boldsymbol{x})\, h(\boldsymbol{x}) \tag{19}$$

For example $f(\boldsymbol{x})$ might be $f(\boldsymbol{x}) = \exp(-x^2 - y^2)$ and $h(\boldsymbol{x}) = \exp(-x^2 - (y-3)^2)$. It is clear that if we rotate both of these functions by $2\pi/3$ and compute their inner product again we will get the same answer

(a) Prove this statement, i.e. prove

$$\langle O_{r_1} f, O_{r_1} h \rangle = \langle f, h \rangle \tag{20}$$

We say that the inner product is invariant under the operations of the group if

$$\langle O_g f, O_g h \rangle = \langle f, h \rangle \tag{21}$$

for all elements of the group.

We recall that the action of a group in a function is given by

$$O_g f(\vec{x}) = f(g^{-1}\vec{x})$$

$$\Rightarrow \langle O_{r_1} f, O_{r_1} h \rangle = \int d^2x \; f^*(g^{-1}x)\, h(g^{-1}x)$$

In this case the representation for the inverse of a rotation in the coordinate space is given by

$$D(r_1^{-1}) = \begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}$$

Therefore, we can do a transformation of coordinates in the previous integral $\vec{x}' = D(r_1^{-1})\,\vec{x}$

$$\Rightarrow \langle O_{r_1} f, O_{r_1} h \rangle = \int d^2x' \, |\det J| \, f^*(\vec{x}')\, h(\vec{x}')$$

but in this case $|\det J| = \left|\det\begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}\right| = 1$

$$\Rightarrow \langle O_{r_1} f, O_{r_1} h \rangle = \int d^2x' \, f^*(\vec{x}')\, h(\vec{x}') = \langle f, h \rangle$$

(b) Let $f_a^{(\mu)}(\boldsymbol{x})$ transform as a row (i.e. row a) of an irreducible representation (i.e. representation (μ)) of the group, i.e.

$$O_g f_a^{(\mu)}(\boldsymbol{x}) = f_b^{\mu}(\boldsymbol{x}) D_{ba}^{(\mu)}(g) \tag{22}$$

Use part (a) to show that

$$\left\langle f_a^{(\mu)}, f_b^{(\nu)} \right\rangle = C^{(\mu)} \delta_{\mu\nu} \delta_{ab} \tag{23}$$

where the coefficient $C^{(\mu)}$ is independent of row, but does depend on the representation. Express $C^{(\mu)}$ using inner products of $f_a^{(\mu)}$.

Use the "colorful" slides from class to heuristically explain this result.

Since the inner product is invariant under representations of the group we have

$$\langle f_a^{(\mu)}, f_b^{(\nu)} \rangle = \frac{1}{N_G} \sum_g \langle O_g f_a^{(\mu)}, O_g f_b^{(\nu)} \rangle = \frac{1}{N_G} \sum_g \langle f_c^{(\mu)}, f_d^{(\nu)} \rangle \sum_{c,d} \left(D_{ca}^{(\mu)}(g)\right)^* D_{db}^{(\nu)}(g)$$

$$= \sum_{c,d} \langle f_c^{(\mu)}, f_d^{(\nu)} \rangle \frac{1}{N_G} \sum_g \left(D_{ca}^{(\mu)}(g)\right)^* D_{db}^{(\nu)}(g) = \sum_{cd} \langle f_c^{(\mu)}, f_d^{(\nu)} \rangle \delta_{cd} \delta_{ab} \delta_{\mu\nu}$$

$$= \sum_c \langle f_c^{(\mu)}, f_c^{(\mu)} \rangle \delta_{ab} \delta_{\mu\nu} = C^{(\mu)} \delta_{\mu\nu} \delta_{ab}$$

where $C^{(\mu)} = \sum_c \langle f_c^{(\mu)}, f_c^{(\mu)} \rangle$

By looking at the slides, we see that any function can be decomposed in a basis determined by the different representations and the rows of each representation. In this sense, we can think of the $f_c^{(\mu)}$ as the projection of said function f in this basis.

(c) Let the Hamiltonian $\mathcal{H}$ commute with the operators of the group

$$O_g \mathcal{H} O_g^{-1} = \mathcal{H} \tag{24}$$

Show that

$$\left\langle f_a^{(\mu)}, \mathcal{H} f_b^{(\nu)} \right\rangle = h^{(\mu)} \delta_{\mu\nu} \delta_{ab} \tag{25}$$

where h^μ is independent of a. Express $h^{(\mu)}$ using inner products of $f_a^{(\mu)}$ and $\mathcal{H}$.

Since in this group we know that the inner product is invariant under the group operations we can see that

$$\langle f_a^{(\mu)}, \mathcal{H} f_b^{(\nu)} \rangle = \frac{1}{N_G} \sum_g \langle O_g f_a^{(\mu)}, O_g \mathcal{H} f_b^{(\nu)} \rangle = \frac{1}{N_G} \sum_g \langle O_g f_a^{(\mu)}, \mathcal{H} O_g f_b^{(\nu)} \rangle$$

Where we used the commutation of $\mathcal{H}$ with g in the last inequality. Now, using the results from b) we see that

$$O_g f_a^{(\mu)} = f_c^{(\mu)} D_{ca}^{(\mu)}(g) \qquad O_g f_b^{(\nu)} = f_d^{(\nu)} D_{db}^{(\nu)}(g)$$

$$\Rightarrow \langle f_a^{(\mu)}, \mathcal{H} f_b^{(\nu)} \rangle = \frac{1}{N_G} \sum_g \langle f_c^{(\mu)}, \mathcal{H} f_d^{(\nu)} \rangle \sum_{c,d} \left((D_{ca}^{(\mu)}(g))^* D_{db}^{(\nu)}(g) \right)$$

$$= \sum_{cd} \langle f_c^{(\mu)}, \mathcal{H} f_d^{(\nu)} \rangle \delta_{cd} \delta_{ab} \delta_{\mu\nu} = \sum_c \langle f_c^{(\mu)}, \mathcal{H} f_c^{(\nu)} \rangle \delta_{ab} \delta_{\mu\nu}$$

$$= h^{(\mu)} \delta_{ab} \delta_{\mu\nu}$$

where we express $h^{(\mu)} = \sum_c \langle f_c^{(\mu)}, \mathcal{H} f_c^{(\mu)} \rangle$.