Dot Product

$$\vec{a}\cdot\vec{b} = (a^i \vec{e}_i)\cdot(b^j \vec{e}_j)$$

$$= a^i b^j (\vec{e}_i \cdot \vec{e}_j) = a^i b^j \delta_{ij} = a_j b^j ,$$

- Contracted indices $a_j b^j$ ↖ j's are "contracted" indicate dot product.

- The dot product is invariant under rotations

$$\underline{a}_j \underline{b}^j = a_i (\mathcal{R}^{-1})^i{}_j (\mathcal{R})^j{}_k b^k = a_i \delta^i{}_k b^k = a_i b^i$$

Cross Product

- For the cross product need the Levi-Civita tensor

$$\varepsilon^{ijk} = \begin{cases} \pm 1 & \text{if } (i,j,k) \text{ is an even/odd perm of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

e.g.

$$\varepsilon^{123} = -\varepsilon^{213} = \varepsilon^{231} = -\varepsilon^{321} = \varepsilon^{312} \ldots$$

$$\varepsilon^{223} = 0$$

The ε^{ijk} symbol and determinants

- ① Consider a 3×3 matrix $m^i{}_j$ to be composed of three row vectors $m^1{}_i$, $m^2{}_j$, $m^3{}_k$, $\mathcal{M} = (\vec{m}^1, \vec{m}^2, \vec{m}^3)$

$$\det \mathcal{M} = \begin{vmatrix} m^1{}_i & \cdots \\ m^2{}_j & \cdots \\ m^3{}_k & \cdots \end{vmatrix}$$

Then

$$\det \mathcal{M} \equiv m^1{}_i\, m^2{}_j\, m^3{}_k\, \varepsilon^{ijk} \equiv \det(\vec{m}^1, \vec{m}^2, \vec{m}^3)$$

- ② The Properties of the determinant:

$\boxed{\text{a)}}$ $\det(\vec{m}^1, \vec{m}^2, \vec{m}^3)$

is a multi-linear function of the rows:

i.e. if $\vec{m}^1 = \vec{a} + \lambda \vec{b}$

constant

$$\det(\vec{a} + \lambda\vec{b}, \vec{m}^2, \vec{m}^3)$$

$$= \det(\vec{a}, \vec{m}^2, \vec{m}^3) + \lambda \det(\vec{b}, \vec{m}^2, \vec{m}^3)$$

We could have used $\vec{m}^2$ or $\vec{m}^3$ instead

b) The determinant is anti-symmetric under interchange of two rows

$$\det(\vec{m}^1, \vec{m}^2, \vec{m}^3) = -\det(\vec{m}^1, \vec{m}^3, \vec{m}^2)$$

Indeed:

$$m^1_i\, m^2_j\, m^3_k\, \varepsilon^{ijk} = -m^1_i\, m^2_j\, m^3_k\, \varepsilon^{ikj}$$

$$= -\, m^1_i\, m^3_k\, m^2_j\, \varepsilon^{ikj}$$

Thus in general

(Eq ★) $$m^l_i\, m^m_j\, m^n_k\, \varepsilon^{ijk} = \det \mathcal{M}\ \varepsilon^{lmn}$$

c) Homework, use Eq ★ to show:

$$\det(AB) = \det(A)\det(B)$$

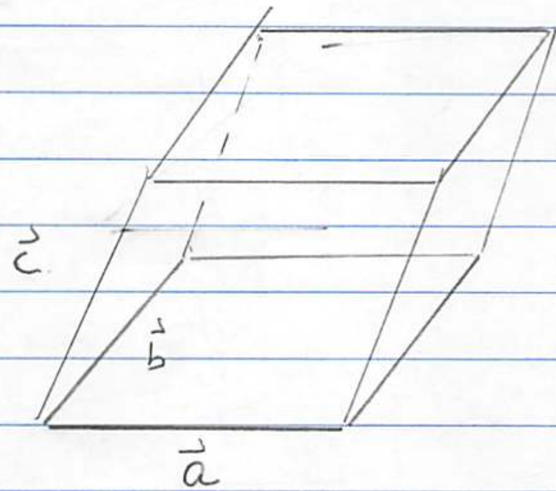
$$\det A^T = \det(A)$$

d) Why is the determinant is important for physics? Take three vectors:

$$\vec{a}, \vec{b}, \vec{c}$$

Then, $\det(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{volume}$

is the volume of the parallel piped spanned by $\vec{a}, \vec{b}, \vec{c}$



(3) The cross product

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{e}_i \left(\varepsilon^{ijk} a_j b_k \right)$$

$$= \vec{e}_i \, (\vec{a} \times \vec{b})^i$$

Thus

$(\vec{a} \times \vec{b})^i = \varepsilon^{ijk} a_j b_k$ = i-th contravariant component of $\vec{a} \times \vec{b}$

The b(ac) - abc rule

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})\,\vec{c}$$

Proof

$$(\vec{a} \times (\vec{b} \times \vec{c}))^i = \varepsilon^{ijk}\, a_j\, (\vec{b} \times \vec{c})_k$$

$$= \varepsilon^{ijk}\, a_j\, \varepsilon_{klm}\, b^l c^m$$

Now analyze and think

$$\varepsilon^{ijk}\, \varepsilon_{klm} = \boxed{\varepsilon^{ijk}\, \varepsilon_{lmk} = \delta^i_l\, \delta^j_m - \delta^i_m\, \delta^j_l}$$

if $k = 3$ and $i, j = 1, 2$ and l, m can be either $1, 2$ or $2, 1$. So l, m is i, j or j, i which is what this expression says

So

$$(a \times (\vec{b} \times \vec{c}))^i = a_j\, b^l c^m\, (\delta^i_l\, \delta^j_m - \delta^i_m\, \delta^j_l)$$

$$= b^i\, (a_m c^m) - (a_l b^l)\, c^i$$

$$= b^i\, (\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})\, c^i$$

Derivative Operations:

$$\boxed{\partial_i \equiv \frac{\partial}{\partial x^i}}$$

grad $= (\nabla \vec{S})_i = \partial_i S$

curl $= (\nabla \times \vec{V})^i = \varepsilon^{ijk} \partial_j V_k$

div $= \nabla \cdot \vec{V} = \partial_i V^i = \partial_x V^x + \partial_y V^y + \partial_z V^z$

laplacian $\nabla \cdot \vec{\nabla} S = \partial_i \partial^i$

The b(ac) − (ab)c rule plays an important role:

$$\nabla \times (\nabla \times \vec{C}) = \vec{\nabla}(\nabla \cdot \vec{V}) - \nabla^2 \vec{C}$$

- Homework: use the b(ac) − (ab)c rule to derive the wave equation

Example of gradient.

- Field of a point charge, $-\vec{\nabla}\frac{1}{r} = \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3}$

Prf

$$-\frac{\partial}{\partial x^i}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial x^i}\frac{1}{(x^\ell x_\ell)^{1/2}}$$

$$= +\frac{1}{2}\frac{1}{(x^\ell x_\ell)^{3/2}}\frac{\partial}{\partial x^i}(x^\ell x_\ell)$$

Use

$$\frac{\partial x^\ell}{\partial x^i} = \delta^\ell_i \quad \text{so} \quad \frac{\partial}{\partial x^i}(x^\ell x_\ell) = \delta^\ell_i x_\ell + x^\ell \delta_{i\ell}$$

$$= 2x_i$$

Thus,

$$\left(-\vec{\nabla}\frac{1}{r}\right)_i = +\frac{1}{\cancel{2}}\frac{1}{(x^\ell x_\ell)^{3/2}}\cancel{2}x_i$$

Use

$$r = (x^\ell x_\ell)^{1/2}$$

$$= \frac{x_i}{r^3}$$

i.e

$$-\vec{\nabla}\left(\frac{1}{r}\right) = \frac{\vec{r}}{r^3} \qquad \text{where } \vec{r} = x_i\vec{e}^{\,i}$$

$$r = \sqrt{x^\ell x_\ell}$$

Helmholtz Theorems:

① If $\vec{\nabla}\cdot\vec{C} = 0$, then there exsists $\vec{D}$ such that:

$$\vec{C} = \vec{\nabla}\times\vec{D}.$$

② If $\nabla\times\vec{C} = 0$, then there exsists a scalar field S such that:

$$\vec{C} = -\vec{\nabla} S,$$

I wont prove it (but see homework) but I will show the converse, i.e.

① $\nabla\cdot(\vec{\nabla}\times\vec{D}) = 0$ and ② $\nabla\times(\vec{\nabla}S) = 0$

Prf.

① $$\partial_i C^i = \partial_i \overbrace{\varepsilon^{ijk}\partial_j D_k}^{(\nabla\times C)^i} = \varepsilon^{ijk}\partial_i\partial_j D_k = 0$$

Because $\varepsilon^{ijk} = -\varepsilon^{jki}$ is antisymmetric while $\partial_i\partial_j = \partial_j\partial_i$ is symmetric, $\partial_x\partial_y - \partial_y\partial_x = 0$

② Similarly, we show $\nabla\times\vec{\nabla}S = 0$

$$\underbrace{\varepsilon^{ijk}\partial_j C_k}_{(\nabla\times C)^i} = \varepsilon^{ijk}\partial_j\partial_k S = 0$$

These are statements of differential forms $ddD = 0$