Electricity and Magnetism + Helmholtz Theorems

$$\nabla \cdot E_{SI} = \rho_{SI} / \varepsilon_0 \qquad\qquad c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$\nabla \times B_{SI} = \mu_0 \vec{J}_{SI} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}_{SI}}{\partial t}$$

$$\nabla \cdot B_{SI} = 0$$

$$-\nabla \times E_{SI} = \frac{\partial B_{SI}}{\partial t}$$

In order to reduce μ_0, ε_0 etc and to make the speed of light more explicit

Define

$$E \equiv \sqrt{\varepsilon_0}\, E_{SI} \qquad\qquad \rho = \frac{\rho_{SI}}{\sqrt{\varepsilon_0}}$$

$$B \equiv \frac{B_{SI}}{\sqrt{\mu_0}} \qquad\qquad \frac{\vec{J}}{c} = \sqrt{\mu_0}\, \vec{J}_{SI}$$

This is known as Heavyside - Lorentz units, and is the set of units I prefer, since the speed of light is explicit.

Then the Maxwell Equations read

a) $\nabla \cdot E = \rho$

sourced

b) $\nabla \times \vec{B} = \frac{\vec{J}}{c} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

c) $\nabla \cdot \vec{B} = 0$

source free

d) $-\nabla \times \vec{E} = \frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

Then we can use the Helmholtz Theorem to good effect in the source free equations

From c) $\nabla \cdot B = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$

From d) $-\nabla \times E = \frac{1}{c}\frac{\partial}{\partial t} \nabla \times A \Rightarrow -\nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \right) = 0$

So there is a ϕ such that:

$$\vec{E} + \frac{1}{c} \partial_t \vec{A} = -\nabla \phi \Rightarrow \vec{E} = -\frac{1}{c} \partial_t \vec{A} - \vec{\nabla} \phi$$

Usually it is easier to work with $(\phi, \vec{A})$ since two of the maxwell equations are automatically satisfied.

E&M and Electrostatics

For $\vec{B} = \vec{J} = 0$

$$\left.\begin{aligned} \nabla \cdot E &= \rho \\ \nabla \times \vec{E} &= 0 \end{aligned}\right\}$$

Using helmoltz since $\nabla \times E = 0$

$$\vec{E} = -\nabla \phi \qquad E_i = -\partial_i \phi$$

Then

$$-\vec{\nabla} \cdot (\vec{\nabla} \phi) = \rho \quad \text{or} \quad \underbrace{-\nabla^2 \phi = \rho}_{\text{poisson eq}} \quad \text{or} \quad \underbrace{-\partial_i \partial^i \phi = \rho}_{\text{same}}$$

We will use the poisson equation to illustrate a number of points throughout the course

E+M and the Stoke's Theorems

① 

$d\vec{a} = d(\text{area})$ vector

$= da\,\vec{n}$

$$\oint_V \nabla\cdot\vec{W} = \oint_{\partial V} d\vec{a}\cdot\vec{W} \qquad \text{Divergence}$$

$$\oint_V \partial_i W^i = \oint_{\partial V} da_i\, W^i$$

② 

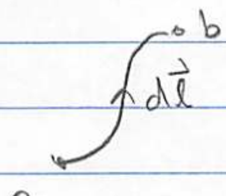
then Curl

$$\int_A d\vec{a}\cdot(\nabla\times\vec{W}) = \oint_{\partial A} \vec{W}\cdot d\vec{l}$$

or

$$\int da_i\, \varepsilon^{ijk}\,\partial_j W_k = \oint dl_i\, W^i$$

③ Finally



$$\int_a^b d\vec{l}\cdot\vec{\nabla}\phi = \phi(x^b) - \phi(x^a)$$

Examples of The Stokes Theorems from EM

- Maxwell Eqns in integral form

d) $$-\nabla \times \vec{E} = \frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$ differential form of faraday law

Integrating over area

$$-\int d\vec{a} \cdot \nabla \times \vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{a}$$

integral form of Faraday law

d) $$-\oint \vec{E} \cdot d\vec{l} = \frac{1}{c}\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{a}$$

Similarly we have for Maxwell equations a), b), c)

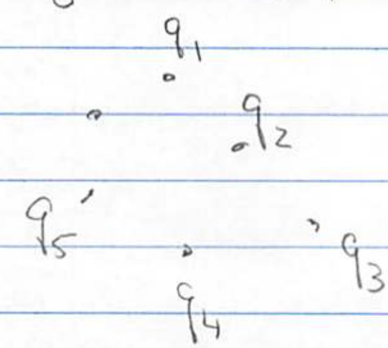
a) $$\oint_{\partial V} \vec{E} \cdot d\vec{a} = \int_V d^3x\, \rho$$

b) $$\oint B \cdot d\vec{l} = \int d\vec{a} \cdot \frac{\vec{j}}{c} + \frac{1}{c}\frac{\partial}{\partial t}\int \vec{E} \cdot d\vec{a}$$

c) $$\oint_{\partial V} B \cdot d\vec{a} = 0$$

Example of Stoke's Theorem and integration by parts

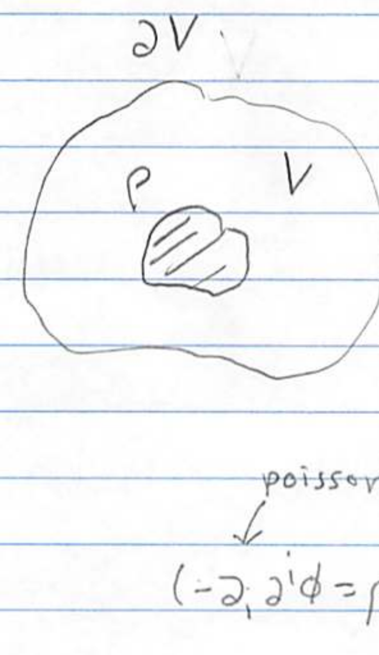
$$U = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{4\pi |\vec{r}_i - \vec{r}_j|}$$



$$u = \frac{1}{2} \sum_i q_i \phi(r_i)$$

For a continuous charge density ρ

$$U = \frac{1}{2} \int d^3r \; \rho(\vec{r}) \, \phi(\vec{r})$$



Want to show

$$u = \frac{1}{2} \int d^3r \; \vec{E} \cdot \vec{E}$$

Proof

poisson eq.

$$u = \frac{1}{2} \int_V d^3\vec{r} \; (-\partial_i \partial^i \phi) \, \phi \qquad (-\partial_i \partial^i \phi = \rho)$$

$$(\partial_i \partial^i \phi) \phi = \partial_i (\partial^i \phi \, \phi) - \partial^i \phi \, \partial_i \phi$$

$$U = \frac{1}{2} \int_V d^3r \; -\partial_i (\partial^i \phi \, \phi) + \frac{1}{2} \int_V \partial^i \phi \, \partial_i \phi$$

Then using the divergence theorem

$$U = \underbrace{\frac{1}{2}\oint_{\partial V} da_i \, (-\partial^i\phi \, \phi)} + \frac{1}{2}\int_V \partial^i\phi \, \partial_i\phi$$

boundary term important in some applications. If V is large enough and $\phi \xrightarrow[r\to\infty]{} 0$ sufficiently quickly, it can be neglected

$$U = \frac{1}{2}\int_V \partial^i\phi \, \partial_i\phi = \frac{1}{2}\int_V \vec{E}\cdot\vec{E}$$

General Rule for integration by parts (IBP)

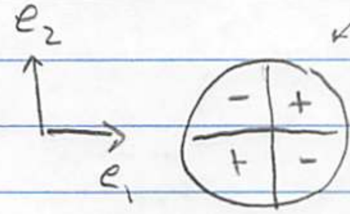
$$\int_V d^3r \, (-\partial_i \, \partial^i\phi)\, \phi = \int_V \partial^i\phi \, \partial_i\phi \; + \text{ bndry terms}$$

move the derivative from one side to other & change sign

I have collected a number of examples from the EM course of IBP in homework.

Tensors

- Example: want to describe the potential from a charge distribution and its orientation. This charge distribution is described by two vectors. Far from this set of charges the potential takes the form



$$\Phi(r) \simeq \frac{1}{4\pi} \frac{1}{2} Q^{ij} \frac{\hat{r}_i \hat{r}_j}{r^3}$$

- Where the quadrupole tensor is

$$Q^{ij} = \int d^3\vec{x}\ \rho(\vec{r})\ (3x^i x^j - r^2 \delta^{ij})$$

- Note this tensor is <u>traceless</u>:

$$\text{trace} = \delta_{ij} Q^{ij} = \int d^3\ \rho(r) \left[3x^i x^j \delta_{ij} - r^2 \delta^{ij} \delta_{ij} \right]$$

Use $\delta^{ij}\delta_{ij} = 3$, $x^i x^j \delta_{ij} = r^2$ to find

$$\text{Trace} = \delta_{ij} Q^{ij} = 0$$

- Rotations rotate each "arm" of the tensor

$$\underline{Q}^{ij} = R^i{}_\ell\, R^j{}_m\, Q^{\ell m} \leftarrow \text{original tensor}$$

$\uparrow$ rotated tensor components

Then, $\overleftrightarrow{Q} = Q'^{ij}\, \vec{e}\,'_i \vec{e}\,'_j$, is the physical tensor and is unchanged by the rotation, since

$$\vec{e}\,'_i\, \vec{e}\,'_j = \vec{e}_l\, \vec{e}_m\, (\mathcal{R}^{-1})^l{}_i\, (\mathcal{R}^{-1})^m{}_j$$