Coordinate Systems - with Cylindrical as example

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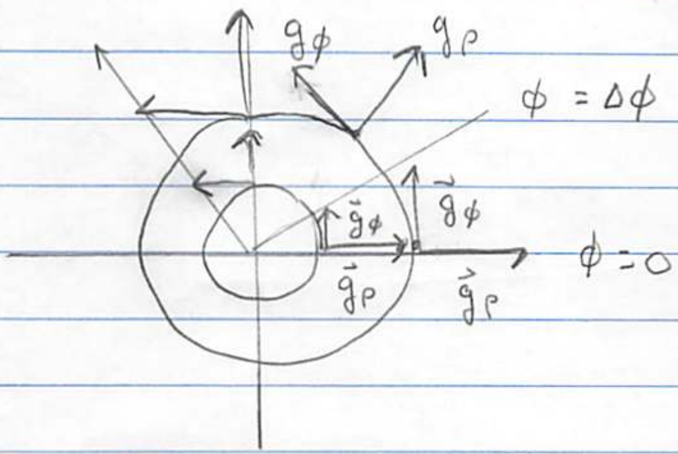


$x = \rho \cos\phi$

$y = \rho \sin\phi$

$z = z$

- The position vector $\vec{s} = x\hat{x} + y\hat{y} + z\hat{z} = x^i \vec{e}_i$
  The coordinates of the new system are denoted $u^a = (u^1, u^2, u^3)$ where a runs from 1, 2, 3, e.g. $u^a = (\rho, \phi, z)$.

- The vector pointing in the ρ direction is

$$\vec{g}_\rho = \frac{\partial \vec{s}}{\partial \rho} = \frac{\partial x^i}{\partial \rho}\vec{e}_i \qquad \vec{g}_\rho = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

while the vector pointing in the ϕ direction is

$$\vec{g}_\phi = \frac{\partial \vec{s}}{\partial \phi} = \frac{\partial x^i}{\partial \phi}\vec{e}_i \qquad \vec{g}_\phi = -\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y}$$

- In general

$$\vec{g}_a = \frac{\partial \vec{s}}{\partial u^a} = \frac{\partial x^i}{\partial u^a}\vec{e}_i \equiv g^i_a \vec{e}_i$$

This is known as the vierbein.

i.e. $g^i_a \equiv \frac{\partial x^i}{\partial u^a}$ $\leftarrow$ It is the i-th cartesian component of the $\vec{g}_a$ coordinate vector.

- In this way the displacement

$$d\vec{s} = d\rho\, \vec{g}_\rho + d\phi\, \vec{g}_\phi + dz\, \vec{g}_z \equiv du^a\, \vec{g}_a$$

$$= du^a \frac{\partial \vec{s}}{\partial u^a}$$

- The vectors are not necessarily orthogonal or orthonormal. The squared displacement is

$$ds^2 = d\vec{s} \cdot d\vec{s} = (\vec{g}_a \cdot \vec{g}_b)\, du^a du^b \equiv g_{ab}\, du^a du^b$$

We have defined the "metric" tensor g_{ab}

$$g_{ab} = \vec{g}_a \cdot \vec{g}_b$$

For this example

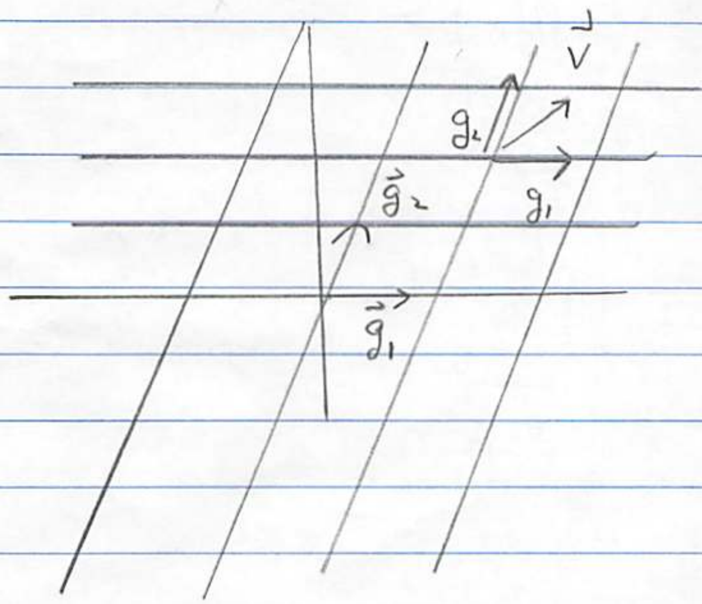
$$\vec{g}_\phi \cdot \vec{g}_\phi = \rho^2 \sin^2\phi + \rho^2 \cos^2\phi = \rho^2$$

$$g_{ab} = \begin{array}{c|c} & \rho \quad \phi \quad z \\ \hline \begin{array}{c} \rho \\ \phi \\ z \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

The metric is

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

② In other coordinate systems the metric tensor will not be orthogonal:



e.g.:

$$x = u^1 + \frac{1}{2}u^2$$

$$y = u^2$$

$$\vec{g}_1 = \hat{x} \qquad \vec{g}_2 = \frac{1}{2}\hat{x} + \hat{y}$$

- If you have a vector $\vec{V}$ you may expand it in terms of $\vec{g}_a$

$$\vec{V} = V^a \vec{g}_a$$

- V^a are the contravariant components (upper)

- $\vec{g}_a$ are the covariant (lower) coordinate basis vectors

- Dot products

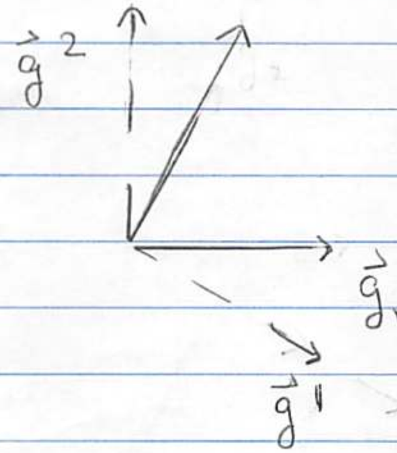
$$\underline{\vec{V}\cdot\vec{W}} = (V^a \vec{g}_a)\cdot(W^b \vec{g}_b) = \underline{V^a W^b g_{ab}} \qquad (\text{Eq} \bigstar)$$

3 In such cases it is useful to introduce a "dual basis" $\vec{g}^a$; defined so that

$$\vec{g}^a \cdot \vec{g}_b = \delta^a{}_b$$

Dual basis definition

Picture



e.g for coordinates on previous page

$$\vec{g}^1 = \hat{x} - \frac{1}{2}\hat{y}$$

$$\vec{g}^2 = \hat{y}$$

It is possible to expand $\vec{v}$ in its dual basis:

$\vec{v} = v_a \vec{g}^a$ with inverse metric tensor $\vec{g}^a \cdot \vec{g}^b = g^{ab}$

Then dot products

$$\vec{v} \cdot \vec{w} = v_a w_b g^{ab} \qquad \text{(Eq ☆☆)}$$

Finally

$$\vec{v} \cdot \vec{w} = v_a \vec{g}^a \cdot w^b \vec{g}_b = v_a w^b \overbrace{\vec{g}^a \cdot g_b}^{\delta^a_b}$$

$$= v_a w^a \qquad \text{(Eq ☆☆☆)}$$

"looks like" its cartesian form

Comparison of ★, ★★ and ★★★ shows

$$W^a = g^{ab} W_b \qquad W_a = g_{ab} W^b$$

Finally we note since

$$W^a = g^{ab} g_{bc} W^c$$

We must have

$$g^{ab} g_{bc} = \delta^a_c$$ i.e. the g^{ab} is the inverse metric tensor

<u>Other notes</u> : Prove me

1. $V^a = \vec{g}^a \cdot \vec{V}$

2. $V_b = \vec{g}_b \cdot \vec{V}$

etc

Volumes in general coordinates

- Recall $\vec{s} = \vec{e}_i \, x^i$

$$\vec{g}_a = \frac{\partial \vec{s}}{\partial u^a} = \vec{e}_i \, \frac{\partial x^i}{\partial u^a} \qquad g^i_a = \frac{\partial x^i}{\partial u^a} \equiv \text{vierbein}$$

The matrix $\partial x^i / \partial u^a = \dfrac{\partial (x^1, x^2, x^3)}{\partial (u^1, u^2, u^3)}$

is

$$(M_0)^i{}_a \equiv \left(\frac{\partial x^i}{\partial u^a}\right) = \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \cdot & \cdot \\ \frac{\partial x^3}{\partial u^1} & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} g^i_1 & g^i_2 & g^i_3 \\ | & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

- The inverse map $u^a(x^i)$ has Jacobian matrix which is inverse of this

$$(M_0^{-1})^a{}_i = \left(\frac{\partial u^a}{\partial x^i}\right) = \frac{\partial (u^1, u^2, u^3)}{\partial (x^1, x^2, x^3)} = \text{Jacobian of inverse map}$$

i.e

$$\frac{\partial x^i}{\partial u^a} \frac{\partial u^a}{\partial x^j} = \delta^i{}_j \qquad \left(\frac{\partial (x^1 x^2 x^3)}{\partial (u^1 u^2 u^3)}\right)\left(\frac{\partial (u^1 u^2 u^3)}{\partial (x^1 x^2 x^3)}\right) = \mathbb{1}$$

- Then $\vec{g}^{\,a} = \dfrac{\partial u^a}{\partial x^i}\vec{e}^{\,i}$ so that $g^a{}_i \equiv \dfrac{\partial u^a}{\partial x^i}$

$$\vec{g}^{\,a}\cdot\vec{g}_b = \delta^a{}_b \qquad \text{(Prove me!)}$$

- Now

$$dV = dx^1\,dx^2\,dx^3$$

$$= \left\| \frac{\partial(x^1 x^2 x^3)}{\partial(u^1 u^2 u^3)} \right\| du^1\,du^2\,du^3$$

absolute value of det M

Find:

$$|\det M_0| = \sqrt{(\det M_0^T)(\det M_0)} = \sqrt{\det g_{ab}} \equiv \sqrt{g}$$

definition

Since

$$M_0^T M_0 = \begin{pmatrix} g_{1i} & \cdot & \cdot \\ g_{2i} & \cdot & \cdot \\ g_{3i} & \cdot & \cdot \end{pmatrix} \begin{pmatrix} g^i{}_1 & g^i{}_2 & g^i{}_3 \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \vec{g}_1\cdot\vec{g}_1 & \vec{g}_1\cdot\vec{g}_2 & \vec{g}_1\cdot\vec{g}_3 \\ \vec{g}_2\cdot\vec{g}_1 & \vec{g}_2\cdot\vec{g}_2 & \vec{g}_2\cdot\vec{g}_3 \\ \vec{g}_3\cdot\vec{g}_1 & \vec{g}_3\cdot\vec{g}_2 & \vec{g}_3\cdot\vec{g}_3 \end{pmatrix} = \begin{pmatrix} g_{ab} \end{pmatrix}$$

Summarizing

$$dV = \sqrt{g}\; du^1 du^2 du^3 \qquad g \equiv \det g_{ab}$$

For the simple case ρ, ϕ, z

$$g_{ab} = \begin{pmatrix} 1 & & \\ & \rho^2 & \\ & & 1 \end{pmatrix} \qquad \sqrt{g} = \rho$$

And

$$dV = \rho\, d\rho\, d\phi\, dz \qquad \text{and} \qquad dV = \underbrace{h_1 h_2 h_3\, du^1 du^2 du^3}_{\text{for othogonal coordinates}}$$

Geometric Picture:

- Take three vectors $\vec{V}_1, \vec{V}_2, \vec{V}_3$

$$\vec{V}_1 \cdot (\vec{V}_2 \times \vec{V}_3) = \begin{vmatrix} v^i_1 & v^i_2 & v^i_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = \text{volume of parallel piped}$$

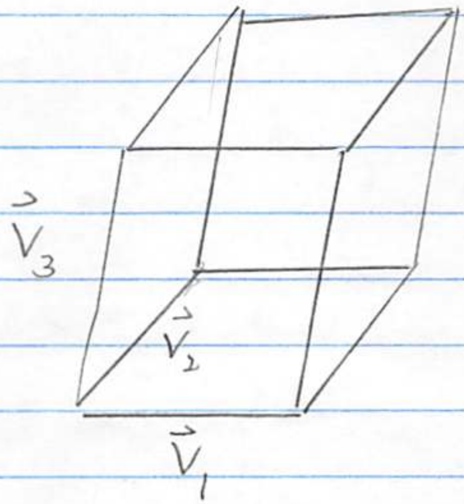
- Then the three vectors spanning a cell are (see picture)

$$du^1\, \vec{g}_1 \,,\quad du^2\, \vec{g}_2 \,,\quad du^3\, \vec{g}_3$$

And the volume of these thee vectors is

$$dV = du^1 du^2 du^3 \begin{vmatrix} g^i_1 & g^i_2 & g^i_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = du^1 du^2 du^3 \,(\vec{g}_1 \cdot (\vec{g}_2 \times \vec{g}_3))$$

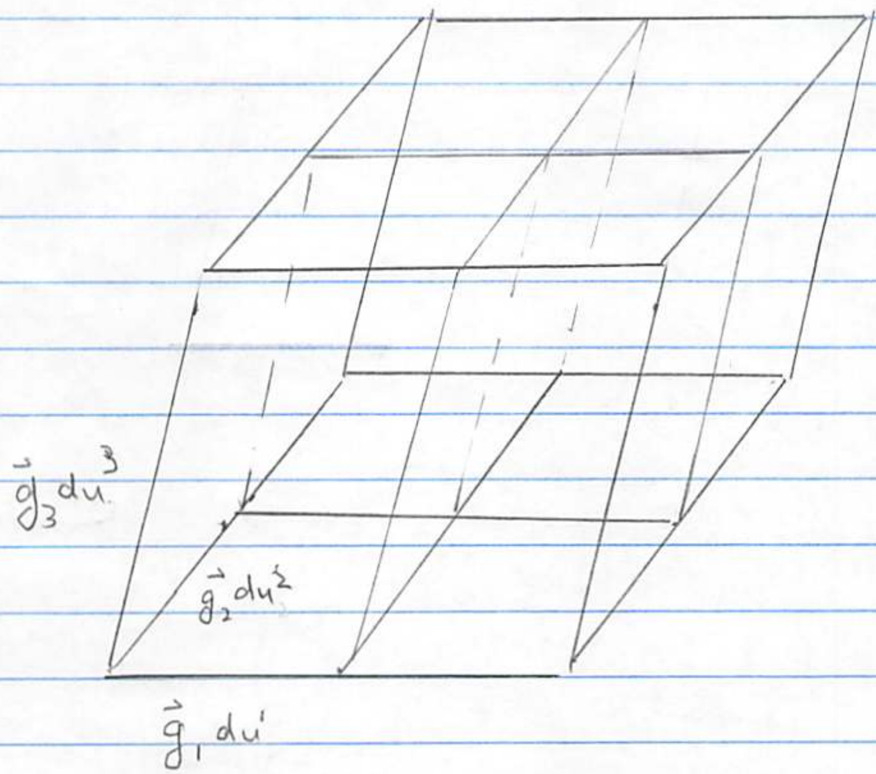
Picture



$$\text{Volume} = \vec{V}_1 \cdot (\vec{V}_2 \times \vec{V}_3)$$

$$= V_1^i V_2^j V_3^k \, \varepsilon_{ijk}$$

$$= \begin{vmatrix} V_1^i & V_2^i & V_3^i \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$$



$$dV = \vec{g}_1 du^1 \cdot (\vec{g}_2 du^2 \times \vec{g}_3 du^3) = \vec{g}_1 \cdot (\vec{g}_2 \times \vec{g}_3) \, du^1 du^2 du^3$$

$$= \begin{vmatrix} g_1^i & g_2^i & g_3^i \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} du^1 du^2 du^3$$

$\leftarrow$ det M_0

Cross Product in General Coordinates

- Need some more notation

$$[abc] = \begin{cases} \pm 1 & \text{for even / odd perms of } 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

For cartesian coordinates $[ijk] = \varepsilon_{ijk}$, but we will now see that for general coordinates

$$\varepsilon_{abc} \equiv \sqrt{g}\,[abc]$$

- In cartesian coordinates $\varepsilon_{ijk} = \vec{e}_i \cdot (\vec{e}_j \times \vec{e}_k)$.
In general coordinates $\underline{\varepsilon_{abc} \equiv \vec{g}_a \cdot (\vec{g}_b \times \vec{g}_c)}$

- With this definition

$$\vec{V}_1 \cdot (\vec{V}_2 \times \vec{V}_3) = V_1^i \vec{e}_i \cdot (V_2^j \vec{e}_j \times V_3^k \vec{e}_k)$$

$$= V_1^i V_2^j V_3^k \underbrace{\varepsilon_{ijk}}_{= e_i \cdot (\vec{e}_j \times \vec{e}_k)}$$

Similarly

$$\vec{V}_1 \cdot (\vec{V}_2 \times \vec{V}_3) = (V_1^a \vec{g}_a) \cdot [(V_2^b \vec{g}_b) \times (V_3^c \vec{g}_c)]$$

$$= V_1^a V_2^b V_3^c \underbrace{\varepsilon_{abc}}_{= \vec{g}_a \cdot (\vec{g}_b \times \vec{g}_c)}$$

• Note

$$\varepsilon_{abc} = \vec{g}_a \cdot (\vec{g}_b \times \vec{g}_c) = \begin{vmatrix} g_a^i & g_b^j & g_c^i \\ \vdots & \vdots & \vdots \end{vmatrix}$$

Up to sign. $\det(\vec{g}_1, \vec{g}_2 \vec{g}_3) = \pm \det(\vec{g}_a, \vec{g}_b, \vec{g}_c)$.
the sign is $[abc]$ = number of Flips to bring abc to 123:

$$\varepsilon_{abc} = \begin{vmatrix} g_1^i & g_2^i & g_3^i \\ \vdots & \vdots & \vdots \end{vmatrix} [abc]$$

by the discussion of last section

$$\underline{\varepsilon_{abc} = \sqrt{g}\,[abc]}$$

• Now we note the following:

$$\underline{(\vec{V} \times \vec{W})_a = \varepsilon_{abc} V^b W^c}$$

$$\vec{X} \cdot (\vec{V} \times \vec{W}) = \varepsilon_{abc} X^a V^b W^c$$

• Finally define

$$\varepsilon^{abc} \equiv g^{aa'} g^{bb'} g^{cc'} \varepsilon_{a'b'c'}$$

Prove me! in homework

$$\underline{\varepsilon^{abc} = \frac{1}{\sqrt{g}} [abc]}$$

So

$$(\vec{V} \times \vec{W})^a = \varepsilon^{abc} V_b W_c$$

$$\vec{V} \times \vec{W} = \varepsilon^{abc} V_b W_c \vec{e}_c \qquad \text{etc}$$