1 Cartesian Tensors and Rotations

1.1 Basics

(a) We will use the Einstein summation convention

$$\boldsymbol{V} = V^1\boldsymbol{e}_1 + V^2\boldsymbol{e}_2 + V^3\boldsymbol{e}_3 = V^i\boldsymbol{e}_i \tag{1.1}$$

Here repeated indices are implicitly summed from $i = 1\ldots 3$, where $1, 2, 3 = x, y, z$ and $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the unit vectors in the x, y, z directions.

(b) Under a rotation of coordinates the coordinates change in the following way

$$\underline{x}^i = R^i_{\ j} x^j = (\mathscr{R})^i_{\ j} x^j \,. \tag{1.2}$$

Here we think of $R^i_{\ j}$ as the i-th,j-th entry of a rotation matrix $\mathscr{R}$, where i labels the rows of $\mathscr{R}$ and j labels the columns of $\mathscr{R}$[1]

(c) Scalars, vectors and tensors are defined by how their components transform

$$S \to \underline{S} = S \,, \tag{1.3}$$

$$V^i \to \underline{V}^i = R^i_{\ j} V^j \,, \tag{1.4}$$

$$T^{ij} \to \underline{T}^{ij} = R^i_{\ \ell} R^j_{\ m} T^{\ell m} \,. \tag{1.5}$$

We think of upper indices (contravariant indices) as row labels, and lower indices (covariant indicies) as column labels. Thus V^i is thought of as column vector

$$V^i \leftrightarrow \begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix} \tag{1.6}$$

labelled by V^1, V^2, V^3 – the first row entry, the second row entry, the third row entry. Contravariant means "opposite to coordinate vectors" $\boldsymbol{e}_i$ (see next item)

(d) Under a rotation of coordinates the basis vectors also transform with

$$\boldsymbol{e}_i \to \underline{\boldsymbol{e}}_i = e_i (\mathscr{R}^{-1})^i_{\ j} \tag{1.7}$$

This transformation rule is how the lower (or covariant) vectors transform. The covariant components of a vector V_i transform as

$$(\underline{V}_1 \underline{V}_2 \underline{V}_3) = (V_1 V_2 V_3)\left(\mathscr{R}^{-1}\right) . \tag{1.8}$$

covariant means "the same as coordinate vectors", *i.e.* with $\mathscr{R}^{-1}$ but as a row. Technicallly this is called the "inverse-transpose representation". For more general transformations, the lower components will follow this "inverse-transpose" rule.

[1] the placement of indices (i.e. up and down) around a matrix is arbitrary. The first index is the row and the second index is column:

$$(\mathscr{R})^i_{\ j} = (\mathscr{R})_{ij} = (\mathscr{R})_i^{\ j} = R^i_{\ j}$$

The indices of the *entries* $R^i_{\ j}$, however, can not be raised or lowered except by multiplying by the metric tensor, δ_{ij}. This distinction is irrelevant here since the metric tensor is so simple, but the distinction will be important for a non-trivial metric like in relativity.

(e) Since $\mathscr{R}^{-1} = \mathscr{R}^T$ there is no need to distinguish covariant and contravariant indices for rotations. This is not the case for more general groups.

(f) With this notation the vectors and tensors (which are physical objects)

$$\underline{\boldsymbol{V}} = \underline{V}^i \underline{\boldsymbol{e}}_i \quad = \quad V^i \boldsymbol{e}_i = \boldsymbol{V} \tag{1.9}$$

$$\underline{\boldsymbol{T}} = \underline{T}^{ij} \underline{\boldsymbol{e}}_i \underline{\boldsymbol{e}}_j \quad = \quad T^{ij} \boldsymbol{e}_i \boldsymbol{e}_j = \boldsymbol{T} \tag{1.10}$$

are invariant under rotations, but the components and basis vectors change.

(g) Vector and tensor components can be raised and lowered with δ^{ij} which forms the identity matrix,

$$\delta^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.11}$$

i.e.

$$V^i = \delta^{ij} V_j \tag{1.12}$$

We note various trivia

$$\delta^i{}_i = 3 \qquad \delta_{ij}\delta^{ij} = 3 \quad \delta_{ij}\delta^{jk} = \delta_i^k \tag{1.13}$$

(h) We also note the various dot products

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \qquad \boldsymbol{e}^i \cdot \boldsymbol{e}^j = \delta^{ij} \qquad \boldsymbol{e}^i \cdot \boldsymbol{e}_j = \delta^i_j \tag{1.14}$$

Using these results, it is straightforward to find the various components

$$v^i = \boldsymbol{e}^i \cdot \boldsymbol{v} \qquad v_i = \boldsymbol{e}_i \cdot \boldsymbol{v} \tag{1.15}$$

The epsilon tensor, determinants, and cross products

(a) The epsilon tensor ϵ^{ijk} is

$$\epsilon^{ijk} = \epsilon_{ijk} = \begin{cases} \pm 1 & \text{for } i,j,k \text{ an even/odd permutation of 1,2,3} \\ 0 & \text{otherwise} \end{cases} \tag{1.16}$$

For example, $\epsilon^{123} = \epsilon^{312} = \epsilon^{231} = 1 = \epsilon_{123} = 1$ while $\epsilon^{213} = -\epsilon^{123} = -1$.

(b) The determinant of matrix is defined using the epsilon tensor. To state the properties of the determinant we first a matrix by its columns, i.e $\vec{M}_1$ is its first column, $\vec{M}_2$ is its second etc

$$\mathbb{M} = (\vec{M}_1, \vec{M}_2, \vec{M}_3) = \begin{pmatrix} M_1^1 & M_2^1 & M_3^1 \\ M_1^2 & M_2^2 & M_3^2 \\ M_1^3 & M_2^3 & M_3^3 \end{pmatrix} \tag{1.17}$$

and notate the determinant as

$$\begin{vmatrix} M_1^1 & M_2^1 & M_3^1 \\ M_1^2 & M_2^2 & M_3^2 \\ M_1^3 & M_2^3 & M_3^3 \end{vmatrix} \equiv M_1^i M_2^j M_3^k \epsilon_{ijk} \tag{1.18}$$

Form this we have the following properties

i) Multilinearity of columns, i.e if for example $\vec{M}_2 = \vec{a} + \lambda \vec{b}$ then

$$\det(\vec{M}_1, \vec{a} + \lambda \vec{b}, \vec{M}_3) \equiv \det(\vec{M}_1, \vec{a}, \vec{M}_3) + \lambda \det(\vec{M}_1, \vec{b}, \vec{M}_3)$$

ii) Antisymmetric under interchange, e.g.

$$\det(\vec{M}_1, \vec{M}_3, \vec{M}_2) \equiv -\det(\vec{M}_1, \vec{M}_2, \vec{M}_3)$$

Since permuting $\vec{M}_1, \vec{M}_2, \vec{M}_3$ and taking the determinant just returns det$\mathbb{M}$ up to a sign (which is set by wether the permuation is even or odd), we have

$$\det(\vec{M}_\ell, \vec{M}_m, \vec{M}_n) \equiv \det(\mathbb{M})\, \epsilon_{\ell mn} \tag{1.19}$$

iii) Eq. (1.19) and other properties allows us to prove

$$\det(\mathbb{M}) = \det(\mathbb{M}^T) \tag{1.20}$$

and

$$\det(\mathbb{AB}) = \det(\mathbb{A})\det(\mathbb{B}) \qquad \det(\mathbb{A}^{-1}) = \frac{1}{\det(\mathbb{A})} \tag{1.21}$$

iv) Everything that was said about determinants using the columns, i.e. that it is multilinear and antisymmetric under interchange, can be said about determinants using the rows. This follows from Eq. (1.20)

(c) The epsilon tensor is used to define cross products and to simplify cross product expressions:

i) The epsilon tensor is useful for simplifying cross products

$$(\boldsymbol{a} \times \boldsymbol{b})_i \equiv \epsilon_{ijk} a^j b^k \tag{1.22}$$

ii) A useful identity is

$$\epsilon^{ijk}\epsilon_{lmk} = \delta^i_l \delta^j_m - \delta^i_m \delta^j_l \tag{1.23}$$

which can be used to deduce the "b(ac) - (ab)c" rule for cross products

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \tag{1.24}$$

iii) The "b(ac) - (ab)c" rule arises a lot in electrodynamics and is essential to deriving the wave equation

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} \tag{1.25}$$

and to identifying the transverse pieces of a vector. For instance the component of a vector $\boldsymbol{v}$, transverse to a unit vector $\boldsymbol{n}$, is

$$-\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{v}) = \boldsymbol{v}_T = -(\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n} + \boldsymbol{v} \tag{1.26}$$

Gradient, Divergence, Curl, Helmholtz Theorems

(a) Derivatives work the same way. $\partial_i \equiv \frac{\partial}{\partial x^i}$. With this notation we have

$$\nabla \cdot \boldsymbol{E} = \partial_i E^i \tag{1.27}$$

$$(\nabla \times \boldsymbol{E})^i = \epsilon^{ijk}\partial_j E_k \tag{1.28}$$

$$(\nabla\phi)_i = \partial_i \phi \tag{1.29}$$

$$(\nabla^2\phi) = \partial_i \partial^i \phi \tag{1.30}$$

$$\tag{1.31}$$

and expressions like

$$\partial_i x^j = \delta^j_i \qquad \partial_i x^i = d = 3 \tag{1.32}$$

In class we used "index-slinging" to show that

$$-\partial_i \frac{1}{r} = \frac{x_i}{r^3} \qquad \text{where} \qquad r = (x^\ell x_\ell)^{1/2} \tag{1.33}$$