Problem 1.  (Goldstein) Oscillations of a bar

A uniform bar of length $\ell$ and mass $m$ is suspended by two equal springs of equilibrium length $b$ and force constant $k$ as shown in the figure below. Determine the normal modes of small oscillations in the plane and the associated frequencies. Sketch the normal modes. Hint: you may wish to consider oscillations which are even (symmetric) and odd (anti-symmetric) with respect to $x \to -x$ separately.
Figure

a) Setup

b) Normal mode with zero frequency

Odd

Displacement at (1) and (2) are along \( \vec{m}_1 \) and \( \vec{m}_2 \)

Normal mode with frequency \( 4 + 2\cos2\theta = \frac{\omega^2}{klm} \)

Odd

Even

frequency, \( \omega^2 = 2\frac{k}{m} \)
Solution: The system is consists of two cm coordinates \( x, y \) and one orientation angle \( \theta \). The kinetic energy of the system is

\[
T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 + \frac{1}{2} I \dot{\theta}^2
\]

(1)

The moment of inertia of a rod is

\[
I = \frac{1}{2} m \ell^2
\]

(2)

The potential energy is expanded around the equilibrium situation, and it will be quadratic in the coordinates. The displacement of the ends of the rod for an angle \( \theta \) and \( x \) and \( y \) o quadratic order reads

\[
\vec{x}_1 = (x_1, y_1) = (x - \theta^2 (\ell/2)^2, y + \theta \ell/2)
\]

(3)

\[
\vec{x}_2 = (x_2, y_2) = (x + \theta^2 (\ell/2)^2, y - \theta \ell/2)
\]

(4)

Notice that (to linear order) the displacement in \( \vec{x}_1 \) and \( \vec{x}_2 \) associated with the cm coordinate \( y \) is even under the reflection symmetry of the problem. But the displacements associated with the bar coordinates \( x \) and \( \theta \) are odd under this symmetry. Thus the potential must take the form:

\[
U = \frac{1}{2} k_{yy} y^2 + \frac{1}{2} (x \theta) \begin{pmatrix} k_{xx} & k_{x\theta} \\ k_{x\theta} & k_{\theta\theta} \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix}
\]

(5)

We will verify this below, and determine the constants.

First let us solve for the normal modes. The equation of motion for \( y \) is

\[
m \ddot{y} = k_{yy} y
\]

(6)

with normal frequency

\[
\omega_1^2 = \frac{k_{yy}}{m}
\]

(7)

The equation of motion of the remaining coordinates takes the form

\[
\begin{pmatrix} m & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = - \begin{pmatrix} k_{xx} & k_{x\theta} \\ k_{x\theta} & k_{\theta\theta} \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix}
\]

(8)

If \( x(t) = u_1 e^{i\omega t} \) and \( \theta(t) = u_2 e^{i\omega t} \) we find a generalized eigenvalue problem:

\[
\omega^2 \begin{pmatrix} m & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} k_{xx} & k_{x\theta} \\ k_{x\theta} & k_{\theta\theta} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

(9)

for the frequencies and displacements \( u_1, u_2 \).

Now we find the spring constants. The potential from the right spring is

\[
\frac{1}{2} k (L - b)^2.
\]

(10)
For the spring displaced from from their equilibrium points by $\vec{x}$ from a point $r$

\[ L = \sqrt{r^2 + \vec{x}^2 + 2\vec{x} \cdot r} \]  
\[ \simeq r + \vec{x} \cdot n + \frac{1}{2} \left( \frac{\vec{x}^2 - (\vec{x} \cdot n)^2}{r} \right) \]  
\[ = r + (\vec{x} \cdot n) + \frac{1}{2} (\vec{x} \cdot m) \]

where $n$ is a unit vector in the direction of $r$. And $m$ is perpendicular to $n$. For the (right) first spring

\[ n_1 = (-\sin(\theta_0), -\cos(\theta_0)) \]  
\[ m_1 = (-\cos(\theta_0), \sin(\theta_0)) \]

While for the left spring

\[ n_2 = (\sin(\theta_0), -\cos(\theta_0)) \]  
\[ m_2 = (\cos(\theta_0), \sin(\theta_0)) \]

So the potential from the springs and gravity reads

\[ U = \frac{1}{2} k(L_1 - b)^2 + \frac{1}{2} k(L_2 - b)^2 + mg y \]

This in general leads to a fairly complicated result of the form

\[ U = \frac{1}{2} k(r - b)^2 + (mg - 2k(r - b) \cos\theta_0) y + \frac{1}{2} k_{yy} y^2 + \frac{1}{2} (x \theta) \begin{pmatrix} k_{xx} & k_{x\theta} \\ k_{x\theta} & k_{\theta\theta} \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix} \]

The potential energy stored in the springs is $U^{(0)}$, while the term linear $U^{(1)}$ vanishes, since by the upward force of the springs $2k(r - b) \cos(\theta_0)$ balances the downward force of gravity $mg$.

It does not seem worth it to record the general $2 \times 2$ matrix here, except in the limit where the mass is light and $r \approx b$, where

\[ \begin{pmatrix} k_{xx} & k_{x\theta} \\ k_{x\theta} & k_{\theta\theta} \end{pmatrix} = k \begin{pmatrix} 2 \sin^2 \theta_0 & \ell \cos(\theta_0) \sin(\theta_0) \\ \ell \cos(\theta_0) \sin(\theta_0) & \frac{1}{2} \ell^2 \cos^2(\theta_0) \end{pmatrix} \]

and

\[ k_{yy} = 2k \cos^2 \theta_0. \]

Then we may use that

\[ I = \frac{1}{12} m \ell^2, \]

The characteristic frequencies of the eigenvalue problem are given by

\[ \det \left[ \omega^2 \begin{pmatrix} m & 0 \\ 0 & \frac{1}{12} m \ell^2 \end{pmatrix} - \begin{pmatrix} k_{xx} & k_{x\theta} \\ k_{x\theta} & k_{\theta\theta} \end{pmatrix} \right] = 0 \]
leading to
\[ \omega^2 = 0, (4 + 2 \cos 2\theta_0) \frac{k}{m} \] (24)

The two modes are
\[ E_0 = \left( -\frac{\ell}{2} \cos \theta_0, \sin \theta_0 \right) \quad E_{4+2 \cos^2 \theta_0} = \left( \ell \sin \theta_0, 6 \cos \theta_0 \right) \] (25)

So to summarize the eigen frequencies are approximately
\[ \frac{\omega^2}{k/m} = 0, 2 \cos^2 \theta_0, 4 + 2 \cos(2\theta_0) \] (26)

Let us analyze the eigenmode with zero frequency. The \( x_1, y_2 \) displacement of the rightmost end of the rod with this mode is
\[ \vec{x}_1 = \frac{\ell}{2} (-\cos \theta_0, \sin \theta_0) = \frac{\ell}{2} m_1 \] (27)

By “rocking” (slowly) back and forth we may leave the length of the springs essentially unchanged. The displacement is along \( m_1 \), which is perpendicular to the spring direction direction \( n_1 \).
Problem 2. (Goldstein) A molecule with a right triangle

The equilibrium configuration of a molecule consists of three identical atoms of mass \( m \) at the vertices of a 45° right triangle connected by springs of equal force constant \( k \). The atoms are constrained to move in the \( xy \) plane.

(a) Determine the zero modes and sketch them.

(b) Consider the oscillations which are orthogonal to the zero modes. Determine the non-zero oscillation frequencies and normal modes for the molecule. Sketch the non-zero normal modes.
Figure

a) Setup

b) Zero modes

\[ \begin{align*}
T_x & \quad T_y \\
R_z &
\end{align*} \]

c) \[ E_+ \]

while
Solution:

(a) There are two translational zero modes and one rotational zero mode. If the vector space of displacements is denoted

$$ \vec{Q} = (x_1, y_1, x_2, y_2, x_3, y_3) $$

(28)

Then the translational zero modes in the $x$ and $y$ directions are

$$ \vec{T}_x = (1, 0, 1, 0, 1, 0) $$

(29)

$$ \vec{T}_y = (0, 1, 0, 1, 0, 1) $$

(30)

To work out the rotational zero modes we need to set up a coordinate system Placing the center of mass at the origin, the coordinates of the atoms are

$$ r_{10}, r_{20}, r_{30} $$

(31)

If we call the long length $6a$ then

$$ r_{10} = (-3a, -a) \equiv n_1 $$

(32)

$$ r_{20} = (3a, -a) \equiv n_2 $$

(33)

$$ r_{30} = (0, 2a) \equiv n_3 $$

(34)

The disturbances drawn in the figure are orthogonal to these vectors. Thus the vector displacement associated with the first atom is

$$ z \times r_{10} = (a, -3a) \equiv m_1 $$

(35)

and similarly

$$ z \times r_{10} = (a, -3a) \equiv m_1 $$

(36)

$$ z \times r_{20} = (a, 3a) \equiv m_2 $$

(37)

$$ z \times r_{30} = (-2a, 0) \equiv m_3 $$

(38)

Thus the rotational zero mode is

$$ \vec{R}_z = a(1, -3, 1, 3, -2, 0) = (m_1, m_2, m_3) $$

(39)

(b) Now we perturb the problem But in general we consider $\vec{Q}$ which lies in a subspace orthogonal to the zero modes. We can also divide $\vec{Q}$ into directions which are even / odd under the reflection symmetries. So out the six numbers in $\vec{Q}$ only three are independent. One could choose these to be $x_1, y_1, x_2$ in a pragmatic fashion. However we are motivated by symmetry, and choose as our three independent coordinates

$$ x_\pm = (x_1 \pm x_2)/2 \quad y_3 $$

(40)
So

\[(x_+ + x_-, y_1), (x_+ - x_-, y_2), (x_3, y_3)\]  

(41)

Requiring that this is orthogonal to the \(T_{x,y}\) and \(R_z\) yields a parameterization of a vector which does not shit the center of mass or cause a rotation

\[
\vec{Q} = ((x_+ + x_-, x_+ - \frac{1}{2}y_3), (x_+ - x_-, -x_+ - \frac{1}{2}y_3), (-2x_+, y_3))
\]  

(42)

A figure below shows the how the three displacements distort the molecule. We label this displacements \(\vec{E}_x\) and \(\vec{E}_y\)

\[
\vec{Q} = x_+ (1, 1, 1, -1, -2, 0) + x_- (1, 0, -1, 0, 0) + y_3 (0, -\frac{1}{2}, 0, -\frac{1}{2}, 0, 1)
\]  

(43)

From the symmetry of the problem the \(\vec{E}_x\) and \(\vec{E}_y\) displacements can mix with each other (they are both even under the reflection symmetry). But because of the symmetry of the problem the \(\vec{E}_+\) (which is odd under the reflection symmetry) can not mix with \(\vec{E}_x\) and \(\vec{E}_y\), and therefore must be an eigenvector. We will verify this below.

The potential energy

\[
U = \frac{1}{2}k(\ell_{12} - \ell_{12}^o)^2 + \frac{1}{2}k(\ell_{31} - \ell_{31}^o)^2 + \frac{1}{2}k(\ell_{32} - \ell_{32}^o)^2
\]  

(44)

where for example

\[
\ell_{ab} = \sqrt{(r_{0a} + r_a - r_{0b} - r_b)^2}
\]  

(45)

and for example

\[
\ell_{12}^o = \sqrt{(r_{01} - r_{02})^2} = 6a
\]  

(46)

Straightforward computer algebra gives

\[
U = \frac{1}{2} \begin{pmatrix} x_+ & x_- & y_3 \end{pmatrix} k \begin{pmatrix} 16 & 0 & 0 \\ 0 & 5 & -3/2 \\ 0 & -3/2 & 9/4 \end{pmatrix} \begin{pmatrix} x_+ \\ x_- \\ y_3 \end{pmatrix}
\]  

(47)

The potential would not have been so simple if we did not use \(x_\pm\) and \(y_3\). Then the kinetic energy in this basis is

\[
\frac{1}{2} m\ddot{\vec{Q}} \cdot \dot{\vec{Q}} = \frac{1}{2} \begin{pmatrix} \dot{x}_+ & \dot{x}_- & \dot{y}_3 \end{pmatrix} m \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{pmatrix} \begin{pmatrix} \dot{x}_+ \\ \dot{x}_- \\ \dot{y}_3 \end{pmatrix}
\]  

(48)
Then we can find the eigen frequencies through a straightforward diagonalization

\[ \det(K - \omega^2 M) = 0 \] (49)

This yields

\[ \omega^2 = \frac{3k}{m}, \frac{2k}{m}, \frac{k}{m} \] (50)

As anticipated, one of the eigen mode only involves \( x_+ \)

\[ \vec{E}_+ = E_{2k/m} = (1, 1, 1, -1, -2, 0) \] (51)

The remaining eigenvectors are superpositions of \( \vec{E}_x \) and \( \vec{E}_y \).
Problem 3.  (Landau) Forced oscillations the easier complex way

(a) Determine the retarded green function of the following equations:

(i) \[ \frac{da}{dt} - i\omega_0 a = 0 \quad (52) \]

(ii) \[ \ddot{x} + \eta \dot{x} = 0 \quad (53) \]

(b) Consider the driven harmonic oscillator

\[ \ddot{x} + \omega_0^2 x = \frac{f(t)}{m} \quad (54) \]

Write it as an equation for \( a = \dot{x} + i\omega x \), and use the Green function of (a) to find the specific solution, \( a(t) \).

(c) Suppose the force approaches zero for \( t \to \pm\infty \). If the oscillator was initially at rest, determine the energy in oscillator as \( t \to \infty \). (You should use the complex variable \( a(t) \) for this calculation.)

(d) Consider the specific force

\[ f(t) = \begin{cases} F_0 & \text{if } -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases} \quad (55) \]

Determine and plot the energy in the oscillator for \( t \to \infty \) as a function of \( \omega_0 \tau \).
(a) (i) For the first equation we try to solve
\[
\left( \frac{d}{dt} - i\omega \right) G(t, t_0) = \delta(t - t_0) \tag{56}
\]
It is a first order differential equation. For \( t > t_0 \) the general solution is
\[
G(t, t_0) = Ae^{i\omega t} \tag{57}
\]
For \( t < t_0 \) the retarded Green function is zero:
\[
G(t, t_0) = 0 \tag{58}
\]
Then integrating Eq. 56 from \( t = t_0 - \epsilon \) to \( t_0 + \epsilon \) gives
\[
G(t_0 + \epsilon, t_0) - G(t_0 - \epsilon, t_0) = 1 \tag{59}
\]
So we may adjust \( A \) so that this (Eq. 59) is satisfied yielding
\[
G(t, t_0) = \theta(t - t_0)e^{i\omega(t-t_0)} \tag{60}
\]
(ii) For the second equation we solve for \( t > t_0 \) and find
\[
G(t, t_0) = A + Be^{-\eta t} \tag{61}
\]
while for \( t < 0 \) the green function is zero. Demanding continuity at \( t = 0 \) of these two solutions we find we find
\[
G(t, t_0) = A(1 - e^{-\eta(t-t_0)}) \tag{62}
\]
To determine the remaining constant we integrate from \( t_0 - \epsilon \) to \( t_0 + \epsilon \) yielding
\[
\int_{t_0-\epsilon}^{t_0+\epsilon} \left( \frac{d^2G}{dt^2} + \eta \frac{dG}{dt} \right) = \int_{t_0-\epsilon}^{t_0+\epsilon} \delta(t - t_0) \tag{63}
\]
yielding
\[
\partial_t G(t, t_0)|_{t_0-\epsilon}^{t_0+\epsilon} + \eta G(t, t_0)|_{t_0-\epsilon}^{t_0+\epsilon} = 1 \tag{64}
\]
So since \( G \to 0 \) as \( t \to t_0 \) we find
\[
\partial_t G(t, t_0) = 1. \tag{65}
\]
This fixes the coefficient of \( A \) in Eq. 62 establishing that
\[
G(t, t_0) = \frac{1}{\eta}(1 - e^{-\eta(t-t_0)})\theta(t - t_0). \tag{66}
\]
(b) We write
\[ \frac{d^2x}{dt^2} + \omega_0^2 x = \frac{d}{dt}(\dot{x} + i\omega x) - i\omega(\dot{x} + i\omega x) \] (67)

Thus the equation of motion is
\[ \frac{da}{dt} - i\omega_o a = \frac{f(t)}{m}, \] (68)

Using the Green function we find
\[ a(t) = \int_{-\infty}^{\infty} dt_0 f(t_0)G(t, t_0) \] (69)
\[ = e^{i\omega_0 t} \int_{-\infty}^{t} dt_0 f(t_0)e^{-i\omega_0 t_0} \] (70)

(c) For the specific force we can integrate
\[ a(t) = e^{i\omega_0 t} \frac{F_0}{m} \int_{0}^{\tau} dt_0 e^{-i\omega_0 t_0} \] (71)
yielding
\[ a(t) = e^{i\omega_0 t} \frac{F_0}{-i\omega_0 m} (1 - e^{-i\omega_0 \tau}) \] (72)

The energy is
\[ E(t, \tau) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2 \] (73)
\[ = \frac{m}{2} |a(t, \tau)|^2 \] (74)
\[ = \frac{2F_0^2}{m\omega_0^2} \sin^2(\omega_0 \tau/2) \] (75)

In the limit of a short force \( \omega_0 \tau \ll 1 \) the impulse is \( F_0 \tau \). So, the velocity after the impulse is \( v = (F_0 \tau)/m \). And the energy in the oscillator just after the impulse is \( 1/2mv^2 = (F_0 \tau)^2/2m \). Expanding our expression in Eq. 75 for \( \omega_0 \tau \ll 1 \), it gracefully approaches \( (F_0 \tau)^2/2m \).
Problem 4. (Likharev) An effective mass

Consider an oscillator with generalized coordinate $q$ and resonant frequency $\omega_0$ with an effective mass which is coordinate-dependent $m_{\text{eff}} = m(1 + \epsilon q^2)$. Calculate the frequency of oscillations using secular perturbation theory, and from an integral given in class for the period of one dimensional systems (see “Motion of 1d systems” online).
Solution:
The Lagrangian is

\[ L = \frac{1}{2} m_{\text{eff}}(q) \dot{q}^2 - \frac{1}{2} m \omega_0^2 q^2 \]  

(76)

where \( m_{\text{eff}}(q) = m(1 + \epsilon q^2) \). The equation of motion is

\[ \partial_t (m(1 + \epsilon q^2) \partial_t q) = -m \omega_0^2 q + m \epsilon \dot{q}^2 \]  

(77)

Here we write the zero-th order solution as

\[ q^{(0)}(t) = A(t) \cos(\omega_0 t + \varphi(t)) \]  

(78)

For a steady state solution (such as this) we can consider a form with \( A = \text{const} \) and \( \varphi = \Delta \omega t \). Or simply

\[ q^{(0)}(t) = A \cos(\omega t) \]  

(79)

with \( \omega = \omega_0 + \Delta \omega \) to be adjusted so that no secular terms appear. The equation of motion without approximation can be written:

\[ \ddot{q} + \omega^2 q + \epsilon \left[ \partial_t (q^2 \partial_t q) - q \dot{q}^2 \right] + (\omega_0^2 - \omega^2) q = 0 \]  

(80)

Approximating \( \omega_0^2 - \omega^2 \approx -2 \omega_0 \Delta - \omega \approx 2 \omega \Delta \omega \) at first order in \( \epsilon \). The mass term is also approximated

\[ \partial_t q^2 \partial_t q - q(\partial_t q)^2 = \frac{A^3}{8} \partial_t \left[ (e^{i\omega t} + e^{-i\omega t})^2 \partial_t (e^{i\omega t} + e^{-i\omega t}) \right] \]

\[ - \frac{A^3}{8} \left[ (e^{i\omega t} + e^{-i\omega t})(\partial_t (e^{i\omega t} + e^{-i\omega t}))^2 \right] \]  

(81)

Or

\[ \partial_t q^2 \partial_t q - q(\partial_t q)^2 = -\frac{A^3 \omega^2}{2} \cos(3\omega t) - \frac{A^3}{2} \omega^2 \cos(\omega t) \]  

(82)

The \( \cos(3\omega t) \) term is not in resonance with the oscillator, and in the zeroth approximation can be neglected. It will of course be necessary in a first approximation to keep this term; it will fix \( q^{(1)} \). Then we find that the equation of motion:

\[ \frac{d^2 q^{(1)}}{dt^2} + \omega^2 q^{(1)} - \left[ 2 \omega_0 \Delta \omega + \frac{\omega^2 A^2}{2} \epsilon \right] q^{(0)} - \frac{A^2}{2} \omega^2 \epsilon \cos(3\omega t) = 0 \]  

(83)

Leading to a frequency shift of

\[ \Delta \omega = -\frac{A^2 \epsilon}{4 \omega_0} \]  

(84)

in order to cancel the secular term. In determining this shift we have ignored the difference between \( \omega_0 \) and \( \omega \) since the whole correction is already first order in \( \epsilon \).

Of course the frequency shift is negative, as the mass shift is \( \propto \epsilon q^2 \), resulting in an increase in mass on average. This lowers the frequency.
We can also use the first order integral to determine the period. Recall that for a Lagrangian
\[ L = \frac{1}{2} m(q) \dot{q}^2 - U(q), \] (85)
the first integral is
\[ E = \frac{1}{2} m(q)^2 + U(q). \] (86)
This may be inverted, determining the time evolution of the system:
\[ t - t_0 = \int_{q_0}^{q} dq \left( \frac{m(q)}{2(E - U(q))} \right)^{1/2} \] (87)
The turning points happens when
\[ E = U(q) = \frac{1}{2} m \omega_0^2 q^2 \] (88)
or
\[ q_\pm = \pm \sqrt{\frac{2E}{m \omega_0^2}} \] (89)
Then we may expand the mass
\[ \sqrt{m(q)} = \sqrt{m(1 + \epsilon q^2)^{1/2}} \simeq \sqrt{m(1 + \frac{1}{2} \epsilon q^2)} \] (90)
Leading to an approximate expression for the period
\[ \tau = 2 \int_{q_-}^{q_+} dq (\sqrt{m/2}) \frac{1}{\sqrt{E - U(q)}} + 2 \int_{q_-}^{q_+} dq (\sqrt{m/2}) \frac{\epsilon q^2}{2 \sqrt{E - U(q)}} \] (91)
Defining a dimensionless integration variable
\[ u = \frac{q}{\sqrt{2E/m \omega_0^2}} \equiv \frac{q}{A} \] (92)
which is the amplitude in units of the maximum we find
\[ \omega_0 \tau = 2 \int_{-1}^{1} du \frac{1}{\sqrt{1-u^2}} + A^2 \frac{\epsilon}{2} \int_{-1}^{1} du \frac{u^2}{\sqrt{1-u^2}} \] (93)
The remaining integrals can be done using the $\beta$ function (or gamma function) yielding $2\pi$ and $\pi/2$ respectively for a total shift
\[ \tau = \frac{2\pi}{\omega_0} (1 + \frac{A^2 \epsilon}{4}) \] (94)
The (angular) frequency in agreement with before
\[ \omega = \frac{2\pi}{\tau} \simeq \omega_0 (1 - \frac{A^2 \epsilon}{4} + \ldots) \] (95)
Problem 5. A non-linear oscillator

An oscillator of mass $m$ and resonant frequency $\omega_0$ has a damping force $F_D = -\beta v^3$ with $\beta > 0$. The motion is initialized with amplitude $a_0$ and no velocity at time $t = 0$.

(a) Define suitable dimensionless variables so that a dimensionless version of the equation reads:

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} + \tilde{x} + \epsilon \left(\frac{d\tilde{x}}{d\tilde{t}}\right)^3 = 0$$

What is the condition on $\beta$ that the non-linear term may be considered small?

(b) If the oscillator starts at $\tilde{t} = 0$ with $\tilde{x} = 1$ with $d\tilde{x}/d\tilde{t} = 0$, use secular perturbation theory to determine approximate behavior of $\tilde{x}(\tilde{t})$. Show in particular that the amplitude decreases as $\tilde{t}^{-1/2}$ at late times. Use Mathematica or other program to determine the exact numerical solution\(^1\) and plot the exact and approximate solution for $\epsilon = 0.3$ up to a time $\tilde{t} = 160$.

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\(^1\)Look up NDSolve and figure it out. I find the following Mathematica advice (parts I and II) by my friend and colleague Mark Alford useful.
Solution:

(a) The initial condition is has scale $a_0$ which defines a length scale. We set $m = \omega_0 = a_0 = 1$ this yields

$$\ddot{x} = x/a_0$$

(97)

$$\ddot{t} = \omega_0 t$$

(98)

$$\ddot{v} = v/(\omega_0 a_0)$$

(99)

The original equation is

$$m\ddot{x} + m\omega_0^2 x + \beta \dot{x}^3 = 0$$

(100)

which after dividing by $m\omega_0^2 a_0$ becomes

$$\frac{d^2 \bar{x}}{dt^2} + \bar{x} + \epsilon \left( \frac{d \bar{x}}{dt} \right)^3 = 0$$

(101)

with

$$\epsilon = \frac{\beta (\omega_0 a_0)^3}{m \omega_0^2 a_0}.$$  

(102)

This is the ratio of the viscous forces $\beta (\omega_0 a_0)^3$ to spring forces $m \omega_0^2 a_0$.

(b) We try a specific form for the zeroth order solution:

$$x^{(0)}(t) = A(t) \cos(-\omega_0 t + \varphi(t))$$

(103)

where $\omega_0 = 1$ in practice. We keep it around for clarity. For simplicity we notate

$$\Omega(t) = -\omega_0 t + \varphi(t)$$

(104)

The full solution is

$$x(t) = x^{(0)}(t) + x^{(1)}(t)$$

(105)

Substituting into the equations we find

$$\ddot{x}^{(0)} + \omega_0^2 x^{(0)} = 2A\omega_0 \sin(\Omega) + 2A\omega_0 \dot{\varphi} \cos(\Omega) + O(\dot{A}, \dot{\varphi})$$

(106)

Similarly for the nonlinear term

$$\epsilon \dot{x}^3 = \epsilon (A \omega_0 \sin(-\omega_0 t + \varphi))^3 + O(\epsilon^2)$$

(107)

$$\approx \frac{\epsilon A^3 \omega_0^3}{-8i} (e^{i\Omega} - e^{-i\Omega})^3$$

(108)

$$= -\frac{\epsilon A^3 \omega_0^3}{4} (\sin(3\Omega) - 3 \sin(\Omega))$$

(109)

So the equation of motion reads

$$\ddot{x}^{(1)} + \omega_0^2 x^{(1)} + \sin(\Omega) \left[ 2A\omega_0 + \frac{3}{4} \epsilon A^3 \omega_0^3 \right] + \cos(\Omega) \left[ 2A\omega_0 \dot{\varphi} - \frac{A^2 \omega_0^3}{4} \sin(3\Omega) \right] = 0$$

(110)
In order to ovoid secular term we choose (recalling that \( \omega_0 \))

\[
\frac{dA}{dt} = -\frac{3}{8} \epsilon A^3 \frac{d\varphi}{dt} = 0 \quad (111)
\]

Solving the first equation is easily solved with the boundary condition that \( A = 1 \) at \( t = 0 \):

\[
\frac{dA}{A^3} = -\frac{3}{8} \epsilon dt . \quad (112)
\]

Integrating we find

\[
A = \frac{1}{(1 + \frac{3}{4} \epsilon t)^{1/2}} \quad (113)
\]

The phase \( \varphi \) is constant, and this constant must be set to 0 in order that there is no initial velocity at \( t = 0 \). Thus our final solution is

\[
x^{(0)} = \frac{1}{(1 + \frac{3}{4} \epsilon t)^{1/2}} \cos(t) \quad (114)
\]

A comparison between the numerical solution (the solid red line) and the analytical form (Eq. 114) is given in Fig. 1
Figure 1: