Problem 1.  Nutation of a Heavy Symmetric Top

Consider a heavy symmetric top with one end point fixed.

(a) Write down the Lagrangian from class. Carry out Routh’s procedure explicitly by Legendre transforming with respect to the the conserved momenta $p_{\psi}$ and $p_{\phi}$. Show that $\theta$ obeys the equation of motion following from the Routhian:

$$I\ddot{\theta} = -\frac{\partial U_{\text{eff}}}{\partial \theta},$$

where

$$U_{\text{eff}} = mg\ell \cos \theta + \frac{(p_{\phi} - p_{\psi} \cos \theta)^2}{2I_1 \sin^2 \theta}.$$  

Also show that

$$\dot{\phi} = \frac{p_{\phi} - p_{\psi} \cos \theta}{I_1 \sin^2(\theta)}.$$  

(b) In class we analyzed the limit when gravitational torque is small to the rotational kinetic energy, $mg\ell/(p_{\psi}^2/I_1) \ll 1$. Take $p_{\phi}/p_{\psi} = r$ with $0 < r < 1$. Within this approximation (known as the fast top approximation), if the energy $E$ is adjusted to the minimum of the effective potential, the tip of the top will slowly precess with

$$\dot{\theta} = 0, \quad \text{and} \quad \dot{\phi} = \frac{mg\ell}{p_{\psi}}.$$  

This is shown in Fig. 1(d) which shows the trajectory of the tip of the top on the sphere.

Now if the energy of the system is slightly larger than the minimum of $U_{\text{eff}}$, describe qualitatively the motion in $\theta$ and $\phi$. For what range in $E$ do the first (a) and second (b) figures describe the top’s motion? Explain. Work in the fast top approximation

![Figure 1](image-url)  

(c) Using the fast top approximation outlined in (b), compute the period of oscillations for a given energy $E$, and determine the precession rate $\dot{\phi}(t)$, and angle $\theta(t)$, as a function of time. What is the average precession rate?
Problem 2. A particle in a magnetic field

(a) Write down the Lagrangian and Hamiltonian for a particle in a magnetic field, \( B(\mathbf{r}) \). Compute the Poisson brackets of velocity:

\[ \{ v_i, v_j \} \]

(b) Prove that the value of any function \( f(q(t), p(t)) \) of coordinates and momenta of a system at a time \( t \) can be expressed in terms of the values of \( p \) and \( q \) at \( t = 0 \) as follows:

\[ f = f_0 + \frac{t}{1!} \{ f_0, H \} + \frac{t^2}{2!} \{ \{ f_0, H \}, H \} + \ldots, \tag{5} \]

where \( f_0 = f(p(0), q(0)) \). Apply this formula to evaluate \( p^2(t) \) for a harmonic oscillator.

(c) Evaluate \( v(t) \) for a particle in a constant magnetic field \( B_0 \) using the results of this problem.

Problem 3. Canonical transformations and Poisson Brackets

Consider an infinitesimal change of coordinates, which is not necessarily canonical:

\[ q \rightarrow Q = q + \lambda \Delta q(q, p), \tag{6} \]
\[ p \rightarrow P = p + \lambda \Delta p(q, p). \tag{7} \]

Show that if the Poisson bracket is to remain fixed under the transformation, i.e.

\[ \{ Q, P \} = 1, \tag{8} \]
\[ \{ P, P \} = 0, \tag{9} \]
\[ \{ Q, Q \} = 0, \tag{10} \]

then there must exist a \( G(q, p) \) which generates the transformation. (Hint recall the following theorem: if a vector field is curl free, \( \nabla \times \mathbf{v} = 0 \) it may be written as a gradient of a scalar function, \( \mathbf{v} = -\nabla \phi \).)

Problem 4. 2d isotropic oscillator

Consider the 2d harmonic oscillator which is isotropic

\[ H = \frac{1}{2} \left( p_1^2 + p_2^2 + (\omega_0 x_1)^2 + (\omega_0 x_2)^2 \right) \tag{11} \]

This is an example of an integrable system, which means if the phase space consists of \( 2n \) generalized coordinates there are \( 2n - 1 \) constants of the motion. We will find and interpret these constants here.

(a) Show that

\[ J_3(\mathbf{r}, \mathbf{p}) = \frac{1}{2} (x_1 p_2 - p_2 x_1) \tag{12} \]

generates rotations in the plane. Why is it constant in time?
(b) Determine the infinitesimal transformation generated by
\[ J_1(r, p) = \frac{1}{2\omega_0} \left( \frac{1}{2} p_1^2 + \omega^2 x_1^2 - \frac{p_2^2}{2} - \frac{1}{2} \omega^2 x_2^2 \right) . \] (13)

Show that the computed transformation leaves the Hamiltonian invariant, and that this implies that \( \dot{J}_1 = \{J_1, H\} = 0 \). Give a physical interpretation of this quantity.

(c) Use the Poisson theorem to deduce a third conserved quantity \( J_2 \):
\[ J_2 = \frac{1}{2\omega_0} (p_1 p_2 + \omega_0^2 q_1 q_2) \] (14)

Determine the associated infinitesimal canonical transformation generated by this conservation law, and verify that it is a symmetry of the Hamiltonian.

(d) We have found three integrals of motion. Using similar manipulations to part (c), one may show that
\[ \{J_i, J_j\} = i\epsilon_{ijk} J_k , \] (15)
and that
\[ \left( \frac{H}{2\omega} \right)^2 = J_1^2 + J_2^2 + J_3^2 \] (16)

Thus any random orbit is selected by choosing \( J_1, J_2, J_3 \) to lie on the surface of a sphere. Describe the motion of the orbit in each of the following limiting cases

(i) \( J_1 = J_2 = 0 \)
(ii) \( J_2 = J_3 = 0 \)
(iii) \( J_1 = J_3 = 0 \)

(e) (Optional:) Consider the 2D oscillator in cylindrical coordinates
\[ L = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{1}{2} \omega_0^2 r^2 \] (17)

Consider a particle in this potential is going around in a circle. At \( t = 0 \) it is on the \( x \) axis, and is then given a small extra push of impulse \( \Delta p \) in the \( y \) direction. Using the integrals of motion explain (without detailed calculation) why the perturbed orbit remains closed.

Problem 5. Phase space

Consider an asymptotically large number of particles \( N \to \infty \) filling (not-necessarily uniformly) a finite region of phase space, i.e. the \( x, p \) plane. Let each particle obey some differential equation which may or may not be of Hamiltonian form
\[ \dot{x} = F_1(t, x, p) , \quad \dot{p} = F_2(t, x, p) . \] (18a) (18b)
The number of particles $dN$ at time $t$ with phase-space coordinates between $x$ and $x + dx$ and $p$ and $p + dp$ is known as the phase-space density $f(t, x, p)$:

$$dN = f(t, x, p) dx dp ,$$

$$= \text{number of particles at time } t \text{ between } x \text{ and } x + dx \text{ and } p \text{ and } p + dp .$$ (19a)

i.e. $f(t, x, p) = dN/dx dp$.

(a) (i) Show that for the differential equation given in Eq. (18) that

$$\frac{\partial f(t, x, p)}{\partial t} + \frac{\partial (f(t, x, p) \dot{x}(t, x, p))}{\partial x} + \frac{\partial (f(t, x, p) \dot{p}(t, x, p))}{\partial p} = 0$$ (20)

(ii) Show that if the particles additionally obey Hamilton’s equations of motion then the phase space density is constant along the world line of a particle, i.e.

$$\frac{d}{dt} f(t, x(t), p(t)) = 0$$ (21)

Compare this proof to the proof given in class that the volume of phase space is constant in time.

Show that the field $f(t, x, p)$ obeys a “wrong-sign” Hamilton equation known as the Liouville equation

$$\frac{\partial}{\partial t} f(t, x, p) + \{ f, H \} = 0$$ (22)

(b) If the phase space distribution at time $t = 0$ is

$$f(0, x, p) = \frac{1}{2\pi \Delta x_0 \Delta p_0} \exp \left[ - \frac{(x - X_0)^2}{2 \Delta x_0^2} - \frac{(p - P_0)^2}{2 \Delta p_0^2} \right]$$ (23)

determine the phase space distribution at later time $t$ for a group of particles in a harmonic oscillator potential.

(c) What is the change with time in the volume, the volume in momentum space, and the volume in phase-space which are occupied by a group of particles which move freely along the $x$-axis? At $t = 0$ the ensemble of particle coordinates are uniformly distributed in the interval $x_0 < x < x_0 + \Delta x_0$ and in the interval $p_0 < p < \Delta p_0$.

(d) Do the same for particles which move along the $x$-axis between two walls $[-L/2, L/2]$. Collisions with the walls are elastic. The particles do not interact with each other.

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$^1$In quantum mechanics we usually multiply this by $\hbar = 2\pi \hbar$ so $dN = f(t, x, p) dx dp$

$^2$This is familiar from quantum mechanics. In the quantum mechanical theory the role of the phase space distribution function is played by the density operator $\rho(t, x, x')$ which satisfies the “wrong-sign” Schrödinger equation $\partial_t \rho = -\frac{i}{\hbar} [\rho, H]$ while other operators obey $\partial_t O = -\frac{i}{\hbar} [O, H]$. 

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