Problem 1. A pendulum on a plane

A pendulum initially swings with amplitude $A$ (not necessarily small) on an incline, of inclination angle $\alpha$ with $\alpha \ll 1$. Determine how the amplitude depends on the inclination angle is slowly increased to $\alpha \sim 1$.

You should find $A_{\text{final}} \propto \frac{1}{(\sin \alpha)^{1/4}}$. 
Problem 2. A tutorial on Fourier Transforms

This is intended as a warm up for a number of problems on waves. Answer all parts as briefly as possible.

Define the fourier transform the physicist way:

\[ F(k) = \int_x e^{-ikx} f(x) \quad f(x) = \int_k e^{ikx} F(k) \]  \hspace{1cm} (1)

Here integrals over wavenumbers \( k \) mean the following

\[ \int_x = \int_{-\infty}^{\infty} dx \quad \int_k \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi} \]  \hspace{1cm} (2)

This is my own little notation that you might try: it is convenient since \( \int_k \int_x e^{ikx} = 1 \). Recall that the integral of a pure phase is a delta function:

\[ \int_x e^{-ikx} = 2\pi \delta(k) \quad \int_k e^{ikx} = \delta(x) \]  \hspace{1cm} (3)

We use

\[ f(x) \leftrightarrow F(k) \hspace{1cm} (4) \]
\[ g(x) \leftrightarrow G(k) \hspace{1cm} (5) \]

to indicate Fourier transform pairs. \( a \) is a postive constant of unit length. \( b \) is a positive constant of unit wavenumber \( k \). \( \epsilon \) is a small real constant

(a) \textbf{(Examples:)} Consider the following most useful Fourier transforms that every serious physicist knows by memory:

\[ \frac{1}{2a} \exp(-|x|/a) \leftrightarrow \frac{1}{1 + (ka)^2} \]  \hspace{1cm} (6)
\[ \frac{1}{a} \text{step}(x/a) \leftrightarrow \text{sinc}(ka) \equiv \frac{\sin(ka/2)}{(ka/2)} \]  \hspace{1cm} (7)
\[ \frac{1}{\sqrt{2\pi a^2}} \exp(-x^2/(2a^2)) \leftrightarrow \exp(-\frac{1}{2}a^2k^2) \]  \hspace{1cm} (8)
\[ \theta(x)e^{-bx} \leftrightarrow \frac{1}{b + ik} \]  \hspace{1cm} (9)

Here \( \text{step}(x) = \theta(x + 1/2) - \theta(x - 1/2) \) (see the first panel of Fig. 1) is the square wave function with integral one. Try to think of a way to remember these. For instance the second one is what comes out of a single slit diffraction of experiment. If you don’t know how to do these integrals try to fix that.

The table can be read either way, with the replacements \( k \rightarrow -x \) and an additional factor of \( 2\pi \), e.g.

\[ \frac{1}{2\pi} \frac{1}{1 + (xb)^2} \leftrightarrow \frac{1}{2b} e^{-|k|/b} \]  \hspace{1cm} (10)
In this case $b$ has units $1/\text{length}$ and is the width in $k$-space.

Prove that the first row of this table holds. Graph $\exp(-|x|/a)$ and its Fourier transform for several values of $\epsilon$, with $\epsilon \equiv 1/a$, and $a \to \infty$ or $\epsilon \to 0$. Show that the Fourier transform of $\exp(-|x|/a)$ is $2\pi$ times a Dirac sequence\(^1\):

$$2\pi \times \left[\frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + k^2}\right] = 2\pi \delta_\epsilon(k)$$

Whenever you see a $\delta$-function, it must be remembered that $\delta(x)$ is shorthand for a Dirac sequence $\delta_\epsilon(x)$. Similarly, the identity

$$\int e^{-ikx} = 2\pi \delta(k)$$

is shorthand for a limiting process where the Fourier integral is cutoff in some way. Here we have explored the cutting it off like this

$$\lim_{\epsilon \to 0} \int e^{ikx} e^{-\epsilon|x|} = 2\pi \delta_\epsilon(k).$$

The Fourier integral can be cutoff in many ways, giving many different realizations of the Dirac sequence.

(b) **(Real/Imaginary/Even/Odd)** Consider the following table:

<table>
<thead>
<tr>
<th>If . . .</th>
<th>then . . .</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$ is real</td>
<td>$F(-k) = (F(k))^*$</td>
</tr>
<tr>
<td>$f(x)$ is imaginary</td>
<td>$F(-k) - F(k)$</td>
</tr>
<tr>
<td>$f(x)$ is even ($f(-x) = f(x)$)</td>
<td>$F(-k) = F(k)$ (i.e. $F$ is even)</td>
</tr>
<tr>
<td>$f(x)$ is odd ($f(-x) = -f(x)$)</td>
<td>$F(-k) = -F(k)$ (i.e. $F$ is odd)</td>
</tr>
<tr>
<td>$f(x)$ is real and even</td>
<td>$F(k)$ is real and even</td>
</tr>
<tr>
<td>$f(x)$ is real and odd</td>
<td>$F(k)$ is imaginary and odd</td>
</tr>
<tr>
<td>$f(x)$ is imaginary and even</td>
<td>$F(k)$ is imaginary and even</td>
</tr>
<tr>
<td>$f(x)$ is imaginary and odd</td>
<td>$F(k)$ is real and odd</td>
</tr>
</tbody>
</table>

Table 1:

Prove the first line and state what this means for the even and odd properties of the real and imaginary parts of $F(k)$. Prove the sixth line as well.

---

\(^1\) A Dirac sequence is any family of functions $\delta_\epsilon(x)$, labelled by a parameter $\epsilon$, which satisfies $\int_{-\infty}^{\infty} \delta_\epsilon(x) = 1$ for any $\epsilon$, and approaches zero for $|x| \neq 0$ and $\epsilon \to 0$. 

3
(c) **(Shifting)** Consider the Fourier transform pair \( f(k) \leftrightarrow F(k) \). We have the following properties

\[
\begin{align*}
  f(x) e^{ik_0 x} & \leftrightarrow F(k - k_0) \quad \text{wavenumber shifting} \quad (14) \\
  f(x - x_0) & \leftrightarrow F(k) e^{-ikx_0} \quad \text{spatial shifting} \quad (15)
\end{align*}
\]

Prove the first of these.

(d) **(Scaling)** Consider the Fourier transform pair \( f(k) \leftrightarrow F(k) \).

\[
\begin{align*}
  f(ax) & \leftrightarrow \frac{1}{|a|} F(k/a) \quad \text{spatial scaling} \quad (16) \\
  \frac{1}{|b|} f(x/b) & \leftrightarrow F(bk) \quad \text{wavenumber scaling} \quad (17)
\end{align*}
\]

Prove the first of these.

(e) **(Derivatives)** Show that

\[
\int_x f(x) = F(k) \bigg|_{k=0},
\]

and show more generally that the moments of \( f(x) \) are related to the Taylor series of \( F(k) \) at the origin by

\[
\int_x f(x)(-ix)^n = \left( \frac{d}{dk} \right)^n F(k) \bigg|_{k=0}
\]

The converse also holds the moments of \( F(k) \) are related to the derivatives of \( f(x) \) at the origin

\[
\int_k F(k)(ik)^n = \left( \frac{d}{dx} \right)^n f(x) \bigg|_{x=0}
\]

(f) **(Analyticity/Asymptotic form)** It follows from Eq. (20), that if a function’s Fourier transform \( F(k) \) falls slower than \( 1/k^n \), then its \( n \)-th derivative will not generically exist. Generally, analytic functions (which have all derivatives) will have a Fourier transform which decreases exponentially at large \( k \). How does Fourier transform of

\[
\frac{1}{1 + (ka)^2}
\]

corroborate the theorem presented in this sub-exercise.

(g) **(Convolutions)** Consider the convolution of two functions

\[
(f * g)(x) \equiv \int_{-\infty}^{\infty} dy \ f(y) \ g(x - y)
\]

Usually the convolution is used to provide a smeared version of the function \( f \). For instance if \( g(x) \) is a normalized narrow gaussian:

\[
g(x) = \frac{1}{\sqrt{2\pi a^2}} \exp\left(-\frac{x^2}{2a^2}\right)
\]

4
then convolution process, just replaces any function \( f(x) \) with a kind of average of all of its neighboring values. The figure below shows a step-like function \( f(x) \) convolved with a gaussian, \( g(x) \). \( g(x) \) has been called the kernel of the convolution.

\[
(f \ast g)(x) \leftrightarrow F(k)G(k) \tag{24}
\]

(i) Working only in coordinate space show that if \( \int y g(y) = 1 \). The integral of \( f \) is unchanged by the convolution process.

(ii) Show that the Fourier transform of the convolution is a product of Fourier transforms.

(iii) By working in fourier space, show using the convolution theorem and Eq. (18) that if \( \int y g(y) = 1 \), then the integral of \( f \) is unchanged by the convolution process.

(iv) Compute the fourier transform of

\[
\left( \frac{\sin(ka/2)}{(ka/2)} \right)^2 \tag{25}
\]

by using the convolution theorem. You can check your result from the its integral in coordinate space.

Describe qualitatively (using the convolution theorem) what are the functions \( B_n(x) \) which are defined by Fourier transform of \( \text{sinc}(ka) \)^n

\[
B_n(x) \leftrightarrow \left( \frac{\sin(ka/2)}{ka/2} \right)^n \tag{26}
\]

These functions \( B_n(x) \) are known as \( B \) splines and are important for numerical work. Note: the higher the \( n \), the faster it falls in \( k \)-space, the smoother the function. Fig. 1 shows the first couple \( B \) splines.
Figure 1: The first four B-splines, $B_1 \ldots B_4$, laid out like two lines of a book. The second one is continuous but has discontinuous derivatives. The fourth one is the cubic bspline, which has continuous second derivatives but discontinuous third derivatives.

(v) Consider the convolution of a smooth function $f(x)$ with a normalized gaussian of width $a$ which is small compared to the scales of $f(x)$. By working in $k$-space, show that

$$(f \ast g)(x) \simeq f(x) + f''(x) \frac{a^2}{2}$$

(27)

Qualitatively, what does $f''(x)$ the term do?

(h) (Correlations) Closely related to the convolution of two functions (but usually rather distinct in physical situations like in statistical mechanics) is the correlation of two functions:

$$\text{Corr}(f, g)(x) \equiv \int dy \, f(x + y) \, g^*(y),$$

(28)

which is relevant when we want to quantify over what range of lengths, $x$, a physical observable $f$ is influenced by value $g$. Often $g$ is a real function and the star is
unnecessary. Show that the correlation function satisfies
\[ \text{Corr}(f, g) \leftrightarrow F(k)(G(k))^* \quad \text{Correlation-Theorem} \quad (29) \]
and thus the fourier transform of *auto-correlation function* is the power spectrum
\[ |F(k)|^2 \]
\[ \text{Corr}(f, f) \leftrightarrow |F(k)|^2 \quad \text{Wiener-Khinchin Theorem} \quad (30) \]
For this exercise you will need to recognize that the Fourier transform of \( g^*(x) \) (what I sometimes call \( G_*(k) \)) is not quite \((G(k))^*\).

(i) **Parseval** Finally prove Parseval’s theorem
\[ \int dx |f(x)|^2 = \int \frac{dk}{2\pi} |F(k)|^2 \quad \text{Parseval’s Theorem} \quad (31) \]
which says that the power can be computed either in coordinate or momentum space.
Problem 3. Equations of motion

(a) From the Euler-Lagrange equations, determine the partial differential equation of motion resulting from the following action:

$$S = \int dt \, dx \left[ \frac{1}{2} \mu (\partial_t q)^2 - \frac{T}{2} (\partial_x q)^2 - V(q) \right]$$

where $V(q) = \lambda q^4$. Also determine the canonical stress tensor for this action.

(b) Determine the partial differential equation of motion resulting from the following action:

$$S = \int dt \, dx \left[ \frac{1}{2} \mu (\partial_t q)^2 - \frac{T}{2} (\partial_x q)^2 - \alpha q \partial_x^4 q \right]$$

(c) Determine the partial differential equation of motion resulting from the following action:

$$S = \int dt \, dx \left[ \frac{1}{2} \mu (\partial_t q)^2 - \frac{T}{2} (\partial_x q)^2 - \alpha (\partial_x^2 q)^2 \right]$$

Compare to part (b) and comment on similarities and differences.
Problem 4. (not on problem set) Normal modes and standing waves

A standing wave is sum of a right moving wave $e^{ikx}$ and a left moving wave, $e^{-ikx}$, often with equal amplitude. Read Likharev sec 5.4 about standing waves if stuck.

In class we found the eigen-frequencies and eigen-modes for a system of $N$ identical particles with masses $m$ connected by identical springs with elastic constants $\kappa$ and separation $a$, with periodic boundary conditions $q_{-N/2} = q_{N/2}$. The general solution is a sum of eigen-vibrations labelled by $k_m = \frac{2\pi m}{Na}$ with $m = -N/2 \ldots N/2 - 1$

\[
q_j(t) = \sum_{m=-N/2}^{N/2-1} A_m \cos(-\omega(k_m)t + k_m ja + \varphi_m) \quad \omega(k_m) = \left[ \frac{4\kappa}{m} \sin^2(ka/2) \right]^{1/2} \quad (32)
\]

(a) Determine the frequencies and eigenmodes of the eigen-vibrations for a system of $N$ identical particles with masses $m$ connected by identical springs with elastic constants $\kappa$ and spatial separation $a$, and find a general solution analogous to Eq. (32). But now take the end points of the chain to be fixed (Fig. 33a).

(b) Repeat (a) when only one end is fixed, and the other end may vibrate freely (Fig. 33b).
Problem 5. (not on problem set) Group velocity of a chain from a continuum theory

(a) Determine the frequencies of the eigen-vibrations of a system of \(2N\) particles, alternating with masses \(m\) and \(M\), connected by springs of elastic constant \(\kappa\) and separation \(a\), and find the general solution as in Eq. (32). The setup is similar to Fig. 33d, but we will assume periodic boundary conditions as in Fig. 33c. Determine the dispersion curve \(\omega(k)\) at small \(k\) to order \(k^3\), also determine the group velocity to order \(k^2\).

You should find

\[
\omega^2(k) = \frac{\kappa}{\hat{\mu}} \pm \sqrt{\left(\frac{\kappa}{\hat{\mu}}\right)^2 - \frac{4\kappa^2}{mM} \sin^2(ka)} \quad (33)
\]

where the reduced mass is

\[
\hat{\mu} = \frac{mM}{m+M}, \quad \frac{1}{\hat{\mu}} = \frac{1}{m} + \frac{1}{M} \quad (34)
\]

(b) When the wavelength of the waves of part (a) is very long, the microscopic details of the discrete model in (a), are unimportant. A continuum theory can reproduce the results of the model in (a), provided the “low energy” constants of the continuum theory are adjusted to match certain physical properties.

Consider the action

\[
S = \int dt \, dx \frac{1}{2} \mu (\partial_t q)^2 - \frac{T}{2} (\partial_x q)^2 - \alpha (\partial_x^2 q)^2 \quad (35)
\]

From the equation of motion you found above, determine the dispersion curve \(\omega(k)\) associated with this action. What should the values of the “low-energy constants” \(\mu\), \(T\) and \(\alpha\) be set to if the continuum action in Eq. (35), is to reproduce the physical observables of the discrete theory of part (a) at small \(k\).