1.4 Motion in a Central Potential

Central potentials $U(r)$ and the Kepler Problem

- We have two bodies with m_1 and $\boldsymbol{r}_1$ and m_2 and $\boldsymbol{r}_2$, and generally take $\boldsymbol{r}_1$ to be the "earth" and $\boldsymbol{r}_2$ and sun. We first switch to center of mass $\boldsymbol{R}$ and relative coordinates $\boldsymbol{r}$

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{M}\,, \tag{1.82}$$

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2\,. \tag{1.83}$$

  with $M = m_1 + m_2$. We have the kinetic energy

$$T = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 \tag{1.84}$$

  where $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass, and thus the Lagrangian is

$$L = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - U(|\boldsymbol{r}|) \tag{1.85}$$

  where $U(|\boldsymbol{r}|)$ is the potential energy of the two particles.

- The overall center of mass motion does not change the orbital dynamics. We can choose $\boldsymbol{R} = \dot{\boldsymbol{R}} = 0$, so that the angular momentum of the center of mass is zero. Then the internal angular momentum is

$$\boldsymbol{L} = \mu\boldsymbol{r}\times\dot{\boldsymbol{r}} \tag{1.86}$$

  $\boldsymbol{L}$ can be chosen to lie along the z axis so that $\boldsymbol{r}$ lies in the x, y plane

$$\boldsymbol{r} = r\,(\cos\phi, \sin\phi, 0) \tag{1.87}$$

  The Lagrangian neglecting the center of mass motion is

$$L = \frac{1}{2}\mu(r^2 + r^2\dot{\phi}^2) - U(r) \tag{1.88}$$

- There are two integrals of motion for the motion in the effective potential:

$$\ell = \mu r^2\dot{\phi}\,, \tag{1.89}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\rm eff}(r,\ell)\,. \tag{1.90}$$

  The effective particle with mass μ moves in the effective potential is

$$V_{\rm eff}(r,\ell) = \frac{\ell^2}{2\mu r^2} + U(r)\,. \tag{1.91}$$

  Given the integrals of motion E and ℓ it is easy to determine $d\phi/d$ and dr/dt. From there it is straightforward to find an equation for $dr/d\phi = \dot{r}/\dot{\phi}$. Integrating $dr/d\phi$ gives the orbit for $r(\phi)$. This integral from (r_1, ϕ_1) to (r, ϕ) is

$$\phi - \phi_1 = \frac{\ell}{\sqrt{2\mu}}\int_{r_1}^{r}\frac{dr/r^2}{\sqrt{E - V_{\rm eff}(r,\ell)}} \tag{1.92}$$

  for an arbitrary potential $U(r)$.

- For the coulomb potential $U = -k/r$, Eq. (1.92) for $r(\phi)$ can be integrated by making the "conformal" substitution

$$u \equiv \frac{1}{r} \qquad du = \frac{dr}{r^2}\,, \tag{1.93}$$

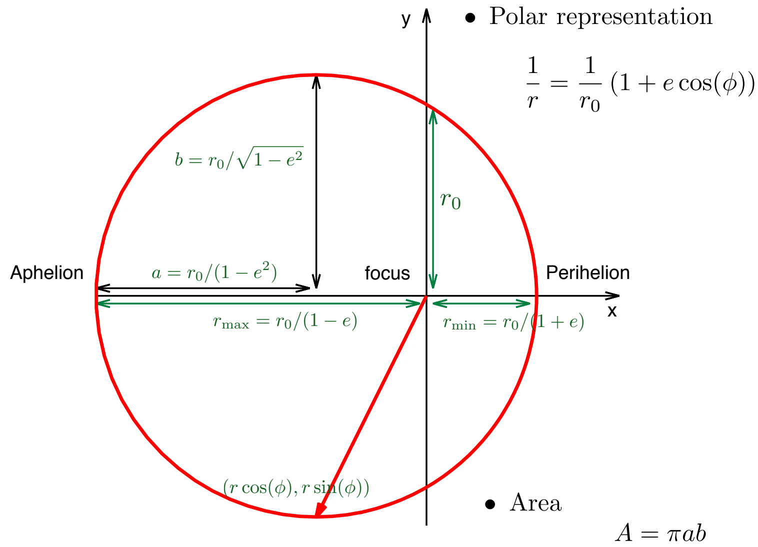


Figure 1.1:

leading to the equation of an ellipse:

$$\frac{1}{r} = \frac{1}{r_0}(1 + e\cos(\phi)) . \tag{1.94}$$

r_0 is known as the lattice rectum (see figure for geometric meaning), and e is known as the eccentricity of the ellipse, which is a measure of how much the orbit deviates from a circle. A convenient summary of the elliptic geometry is given in Fig. 1.1

The parameters of the ellipse r_0 and e are determined by the integrals of motion, E and ℓ. The lattice rectum is determined by the angular momentum, $r_0 = \ell^2/\mu k$. The eccentricity e is determined by the excitation energy above the minimum of $V_{\rm eff}$ (with fixed ℓ). More explicitly $e = \sqrt{1 + E/\epsilon_0}$, with $\epsilon_0 = \ell^2/2\mu r_0^2$. When the energy of the orbit is at its minimum, $E = V_{\rm min} = -\epsilon_0$, then the eccentricity is zero and the radius is constant, i.e. the orbit is circular.

- The Coulomb potential has a characteristic scale $r_0 \sim \ell^2/\mu k$ when the potential k/r_0 and kinetic $\ell^2/\mu r_0^2$ are the same order of magnitude. Indeed, for a circular orbit of radius r_0, one shows by freshman physics that the radius is determined by the angular momentum, $r_0 = \ell^2/\mu k$. For such a circular orbits the kinetic energy is $\epsilon_0 \equiv \ell^2/2\mu r_0$ and is minus-half the potential $U = -k/r_0 = -2\epsilon_0$. The total energy (kinetic+potential) is $E = -\epsilon_0$ where

$$\epsilon_0 \equiv \frac{\ell^2}{2\mu r_0^2} = \frac{k}{2r_0} , \tag{1.95}$$

which explains the notation for the parameters in the previous item.

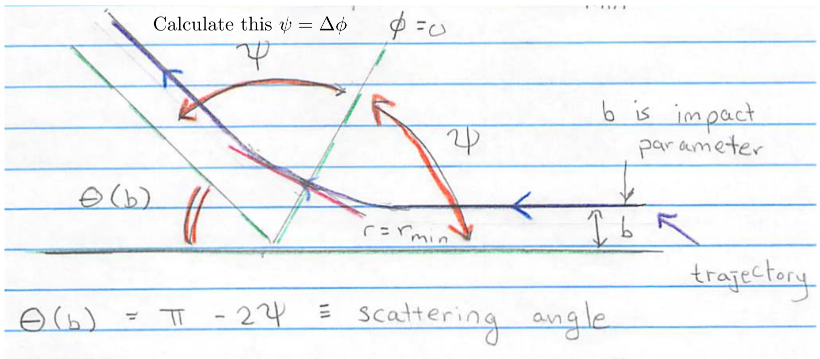
- For the Newton potential $U = -k/r$ and the spherical harmonic oscillator $U = \frac{1}{2}kr^2$ the orbits are closed (Bertrand's theorem). For no other central potentials are the orbits closed. The closed orbits are a consequence of an additional symmetry which we will discuss later.

Cross sections and scattering

- When considering the scattering problem we are interested in computing the scattering angle θ (the angle of deflection) for given energy E and impact parameter b. Here the impact parameter b is the transverse distance at large r from the target and is another way to record the angular momentum. At larger r the velocity is constant, $E = \frac{1}{2}mv^2$, and the angular momentum is

$$\ell = mvr\sin\theta = mvb = \sqrt{2mE}b \tag{1.96}$$

- The scattering angle $\theta(b)$ is shown below:



A particle comes in with impact parameters b (or angular momentum ℓ) and energy E, and is deflected by angle $\theta(b, E)$. From our mechanical perspective we find it easiest to compute the change in the angle ϕ as the particle propagates from its distance of closest approach $r_{\rm min}$ up to infinity. This is (the second) angle ψ in the figure above. It is related to $\theta(b, E)$ by simple geometry.

$$\theta(b) = \pi - 2\psi \,. \tag{1.97}$$

We have from Eq. (1.92)

$$\Delta\phi = \psi = \frac{\ell}{\sqrt{2m}} \int_{r_{\rm min}}^{\infty} \frac{dr/r^2}{(E - V_{\rm eff}(r))^{1/2}} \,. \tag{1.98}$$

For the Coulomb problem $U = k/r$ this integration is straightforward with the substitution $u = 1/r$, and yields $\tan(\psi)$ and since $\psi = \pi/2 - \theta/2$

$$\cot(\theta/2) = \frac{2Eb}{k} \,. \tag{1.99}$$

- The scattering problem is usually phrased in terms of cross section:

  (i) Consider a beam of particles of luminosity $\mathscr{L}$. $\mathscr{L}$ is the number of particles crossing the target per area per time, and is also called the incident flux or intensity.

  (ii) The number of incoming particles which scatter per time $d\Gamma$ with impact parameter between b and db is $d\Gamma = \mathscr{L} 2\pi b |db|$. We put absolute values because we think of db as an positive interval.

  (iii) The number of incoming particles per time (or rate $d\Gamma$) which then end up at in ring of solid angle $d\Omega = 2\pi \sin(\theta)|d\theta|$ per time is

  $$d\Gamma = \mathscr{L} \frac{b}{\sin\theta} \frac{|db|}{|d\theta|} \, d\Omega \,. \tag{1.100}$$

  So the scattering rate per solid angle is

  $$\frac{d\Gamma}{d\Omega} = \mathscr{L} \frac{b}{\sin\theta} \frac{|db|}{|d\theta|} \,. \tag{1.101}$$

  The cross section is by definition the scattering rate divided by the incident flux

  $$\frac{d\sigma}{d\Omega} \equiv \frac{1}{\mathscr{L}} \frac{d\Gamma}{d\Omega} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| \,. \tag{1.102}$$

  (iv) The cross section has units of area and gives a measure of the effective size of the target. It is usually measured in barns, $1\,\text{barn} = 10^{-24}\,\text{cm}^2$.

- For the Coulomb problem, we can different $d\theta/db$ (Eq. (1.99)) and use it in Eq. (1.102) to determine the Rutherford cross section
$$\frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)} \sim \frac{1}{\theta^4}\,, \tag{1.103}$$
which is inversely proportional to $1/\theta^4$ at small angles.