1 Basic Mechanics

1.1 Newtonian mechanics a brief review

Momentum and Center of Mass

- Newton's equations of motion for a system of particles reads

$$\frac{d\boldsymbol{p}_a}{dt} = \boldsymbol{F}_a \tag{1.1}$$

where $a = 1 \ldots N$ labels the particles. Here $\boldsymbol{p}_a = m_a \boldsymbol{v}_a$. We usually divide up the forces on the a-the particle into external forces acting on the system from outside, and internal forces acting between pairs of particles:

$$\boldsymbol{F}_a = \underbrace{\boldsymbol{F}_a^{\text{ext}}}_{\text{external forces}} + \underbrace{\sum_{b \neq a} \boldsymbol{F}_{ab}}_{\text{internal forces}} . \tag{1.2}$$

Here

$$\boldsymbol{F}_{ab} \equiv \text{ Force on particle } a \text{ by } b\,, \tag{1.3}$$

and of course we have Newton's equal and opposite rule

$$\boldsymbol{F}_{ab} = -\boldsymbol{F}_{ba}\,. \tag{1.4}$$

- Summing over the particles we find (after using Eq. (1.4)) that the internal forces cancel and the total change in momentum per time is the sum of external forces

$$\frac{d\boldsymbol{P}_{\text{tot}}}{dt} = \boldsymbol{F}_{\text{tot}}^{\text{ext}} \tag{1.5}$$

where $\boldsymbol{P}_{\text{tot}} = \sum_a \boldsymbol{p}_a$ and $\boldsymbol{F}_{\text{tot}}^{\text{ext}} = \sum_a \boldsymbol{F}_a^{\text{ext}}$. If there are no external forces then $\boldsymbol{P}_{\text{tot}}$ is constant

- The velocity of the center of mass is

$$\boldsymbol{v}_{\text{cm}} = \frac{\boldsymbol{P}_{\text{tot}}}{M_{\text{tot}}} = \frac{1}{M_{\text{tot}}} \sum_a m_a \boldsymbol{v}_a\,. \tag{1.6}$$

The position of the center of mass (relative to an origin O) is

$$\boldsymbol{R}_{\text{cm}} = \frac{1}{M_{\text{tot}}} \sum_a m_a \boldsymbol{r}_a\,. \tag{1.7}$$

Angular momentum:

- Angular momentum is defined with respect to a specific origin O (i.e. $\boldsymbol{r}_a$ depends on O) which is not normally notated

$$\boldsymbol{\ell}_{a,O} \equiv \boldsymbol{\ell}_a \equiv \boldsymbol{r}_a \times \boldsymbol{p}_a\,. \tag{1.8}$$

It evolves as

$$\frac{d\boldsymbol{\ell}}{dt} = \boldsymbol{r}_a \times \boldsymbol{F}_a \tag{1.9}$$

- The total angular momentum $\boldsymbol{L}_{\rm tot} = \sum_a \boldsymbol{\ell}_a$ changes due to the total *external* torque

$$\frac{d\boldsymbol{L}_{\rm tot}}{dt} = \boldsymbol{\tau}^{\rm ext}_{\rm tot} \,, \tag{1.10}$$

where $\boldsymbol{\tau}^{\rm ext}_{\rm tot} = \sum_a \boldsymbol{r}_a \times \boldsymbol{F}^{\rm ext}_a$ were we have generally assumed that the internal forces are radially directed $\boldsymbol{F}_{ab} \propto (\boldsymbol{r}_a - \boldsymbol{r}_b)$

- The angular momentum depends on the origin O. Writing the position of the particle relative to the center of mass as $\Delta\boldsymbol{r}_a$, i.e.

$$\boldsymbol{r}_a = \boldsymbol{R}_{\rm cm} + \Delta\boldsymbol{r}_a \,, \tag{1.11}$$

the angular momentum of the system about O is

$$\boldsymbol{L}_O = \underbrace{\boldsymbol{R}_{\rm cm} \times \boldsymbol{P}_{\rm tot}}_{\text{Ang-mom of center of mass about } O} + \underbrace{\sum_a \Delta\boldsymbol{r}_a \times \boldsymbol{p}_a}_{\text{Ang-mom about the cm}} \,. \tag{1.12}$$

Energy

- Energy conservation is derived by taking the dot product of $\boldsymbol{v}$ with $d\boldsymbol{p}/dt$. We find that the change in kinetic energy (on the a-the particle) equals the work done (on the a-particle).

$$\frac{1}{2} m_a v_a^2(t)\Big|_{t_1}^{t_2} = W_a \tag{1.13}$$

where the work is

$$W_a = \int_{\boldsymbol{r}_a(t_1)}^{\boldsymbol{r}_a(t_2)} \boldsymbol{F}_a \cdot {\rm d}\boldsymbol{r}_a \tag{1.14}$$

- *Potential Energy.* For conservative forces the force can be written as (minus) the gradient of a scalar function which we call the potential energy

$$\boldsymbol{F}_a = -\nabla_{\boldsymbol{r}_a} U \tag{1.15}$$

Consider the potential energy U_{12} between particle 1 and 2. Since the force is equal and opposite

$$\boldsymbol{F}_{12} = -\nabla_{\boldsymbol{r}_1} U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2) = +\nabla_{\boldsymbol{r}_2} U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\boldsymbol{F}_{21} \tag{1.16}$$

and this is used to conclude that interaction potential between two particles is of the form

$$U^{\rm int}_{12} = U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) \tag{1.17}$$

Typically we divide up the potential into an external potential and the internal ones

$$U(\boldsymbol{r}_a) = U^{\rm ext}(\boldsymbol{r}_a) + \frac{1}{2} \sum_{ab, a\neq b} U^{\rm int}_{ab}(\boldsymbol{r}_a, \boldsymbol{r}_b) \tag{1.18}$$

The sum over the internal potentials comes with a factor of a half because the energy between particle-1 and particle-2 is counted twice in the sum, e.g. for just two particles

$$U^{\rm int}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{2} \left(U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) + U(|\boldsymbol{r}_2 - \boldsymbol{r}_1|) \right) \,. \tag{1.19}$$

- *Energy.* The total energy is

$$E = \sum_a \frac{1}{2} m_a v_a^2 + U^{\text{ext}}(\boldsymbol{r}_a) + \frac{1}{2} \sum_{ab, a\neq b} U_{ab}^{\text{int}}(\boldsymbol{r}_a, \boldsymbol{r}_b) \tag{1.20}$$

  and is constant if there are no non-conservative forces.

  If there are non-conservative forces then

$$E(t_2) - E(t_1) = W_{\text{NC}} \tag{1.21}$$

  where the work done by the non-conservative forces is $W_{NC} = \sum_a \int \boldsymbol{F}_a^{NC} \cdot \mathrm{d}\boldsymbol{r}_a$

- It is convenient to measure velocities relative to the center of mass

$$\boldsymbol{v}_a = \boldsymbol{v}_{\text{cm}} + \Delta \boldsymbol{v}_a \tag{1.22}$$

  where $\Delta \boldsymbol{v}_a = \dot{\Delta} \boldsymbol{r}_a$, then the kinetic energy

$$K = \underbrace{\frac{1}{2} M_{\text{tot}} v_{\text{cm}}^2}_{\text{KE of center-mass}} + \underbrace{\sum_a \frac{1}{2} m_a \Delta v_a^2}_{\text{KE relative to center-mass}} \tag{1.23}$$

Galilean invariance:

- Consider newtons laws then for an isolated system of particles

$$\frac{d\boldsymbol{p}_a}{dt} = \boldsymbol{F}_a \tag{1.24}$$

where $\boldsymbol{F}_a = -\nabla_{\boldsymbol{r}_a} U$ with

$$U = \frac{1}{2} \sum_{ab, a\neq b} U_{ab}^{\text{int}}(|\boldsymbol{r}_a - \boldsymbol{r}_b|) \tag{1.25}$$

Here the space-time coordinates are measured by an observer O with origin.

Then consider an observer O' moving with *constant* velocity $-\boldsymbol{u}$ relative to O. The "new" coordinates (those measured by O') are related to the old coordinates via a Galilean boost

$$\boldsymbol{r}_a \to \boldsymbol{r}'_a = \boldsymbol{r}_a + \boldsymbol{u}t \tag{1.26}$$
$$t \to t' = t \tag{1.27}$$

The potential which only depends on $\boldsymbol{r}_a - \boldsymbol{r}_b$ is independent of the shift. The observer measures

$$\boldsymbol{v}_a \to \boldsymbol{v}'_a = \boldsymbol{v}_a + \boldsymbol{u} \tag{1.28}$$
$$\boldsymbol{p}_a \to \boldsymbol{p}'_a = \boldsymbol{p}_a + m_a \boldsymbol{u} \tag{1.29}$$

The equations of motion for observer O' are unchanged

$$\frac{d\boldsymbol{p}'_a}{dt'} = \boldsymbol{F}'_a \qquad \boldsymbol{F}' \equiv \nabla_{\boldsymbol{r}'} U(|\boldsymbol{r}'_a - \boldsymbol{r}'_b|) \tag{1.30}$$

1.2 The action and the Euler Lagrange equations

- The action

$$S[\boldsymbol{r}(t)] = \int_{t_1}^{t_2} dt L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \tag{1.31}$$

takes an arbitrary path $\boldsymbol{r}(t)$ (which may not satisfy the EOM) and returns a number. It is called a *functional*.

- The action principle says that the path $\boldsymbol{r}(t)$ that satisfies the EOM (sometimes called the the classical or "on-shell" path) is an extremum the action[1]. This means that if we replace the on-shell path $\boldsymbol{r}(t)$ with

$$\boldsymbol{r}(t) \to \boldsymbol{r}(t) + \delta\boldsymbol{r}(t) \tag{1.32}$$

for an arbitrary (small) function $\delta\boldsymbol{r}(t)$ that vanishes near t_1 and t_2 then the action is unchanged

$$S[\boldsymbol{r}(t) + \delta\boldsymbol{r}(t)] = S[\boldsymbol{r}(t)] \qquad \text{when } \boldsymbol{r}(t) \text{ is "on-shell", i.e. satisfies the EOM} \tag{1.33}$$

- Generally we define

$$\delta S[\boldsymbol{r}(t), \delta\boldsymbol{r}(t)] \equiv S[\boldsymbol{r}(t) + \delta\boldsymbol{r}(t)] - S[\boldsymbol{r}(t)] \tag{1.34}$$

and note that $\delta S[\boldsymbol{r}, \delta\boldsymbol{r}]$ depends on both the path and the variation. The requirement that $\delta S = 0$ determines the equation of motion. You should be able to prove that when $\delta S = 0$ for an arbitrary variation, the equations of motion are (in 1d)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{1.35}$$

- For a general set of coordinates $q^A = 1 \ldots N$ the equations of motion take the same form:

$$\delta S \equiv S[q(t) + \delta q(t)] - S[q(t)] = 0 \tag{1.36}$$

to first order in an arbitrary $\delta q(t)$. This leads to N equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} = \frac{\partial L}{\partial q^A} \qquad A = 1 \ldots N \tag{1.37}$$

we call

$$p_A = \frac{\partial L}{\partial \dot{q}^A} \equiv \text{the canonical momentum conjugate to } q^A \tag{1.38}$$

$$F_A = \frac{\partial L}{\partial q^A} \equiv \text{the generalized force associated with } q^A \tag{1.39}$$

- If a coordinate q^A does not appear in the Lagrangian (but of course $\dot{q}^A$ does or it wouldn't appear at all), the variable is called *cyclic*. For a cyclic coordinate we have from the Euler Lagrange equations (Eq. (1.37))

$$\frac{dp_A}{dt} = 0 \tag{1.40}$$

i.e. p_A is a constant of the motion.

The hamiltonian function

- The hamiltonian (or energy) function (sometimes called the "first integral") is

$$h(q, \dot{q}, t) = p\dot{q} - L(q, \dot{q}, t) = \frac{\partial L}{\partial \dot{q}}\dot{q} - L(q, \dot{q}, t) \tag{1.41}$$

and obey the equation of motion

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}\,. \tag{1.42}$$

$h(q, \dot{q}, t)$ is therefore constant if L does not depend explicitly on time.

[1] Sometimes for clarity we will put a bar, e.g., $\underline{\boldsymbol{r}}(t)$ to indicate that this path is on-shell, i.e. that it satisfies the EOM

- If more then one coordinate is involved then

$$h(q^A, \dot{q}^A, t) = \sum_A p_A \dot{q}^A - L \tag{1.43}$$

$$= \frac{\partial L}{\partial \dot{q}^A} \dot{q}^A - L \tag{1.44}$$

where we have and will *from now on* follow the summation convention, where repeated indices are summed over.

- We will distinguish the hamiltonian function $h(q, \dot{q}, t)$, which is a function of q, $\dot{q}$, and t, from the Hamiltonian $H(p, q, t)$ which is a function of q and p and t through the Legendre transform (more later). Thus $p_A(q, \dot{q}, t)$ in the hamiltonian function (Eq. (1.43)) is a function of the q and the $\dot{q}$, while in the Hamiltonian the $\dot{q}$ is a function of q and p.

- For a rather general Lagrangian

$$L = \frac{1}{2} a_{ij}(q)\, \dot{q}^i \dot{q}^j + b_i(q) \dot{q}^i - U(q)\,, \tag{1.45}$$

(which is the form of the Lagrangian for a particle in a magnetic field or gravity) the hamiltonian function is

$$h(\dot{q}, q, t) = \frac{1}{2} a_{ij}(q)\, \dot{q}^i \dot{q}^j + U(q) \tag{1.46}$$

The fact that the hamiltonian function is independent of b_i is closely related to the fact that magnetic fields do no work.

The period of one dimensional motion

- For one dimensional Lagrangian's of the form

$$L = \frac{1}{2} m(q)\, \dot{q}^2 - V_{\text{eff}}(q) \tag{1.47}$$

The first integral is

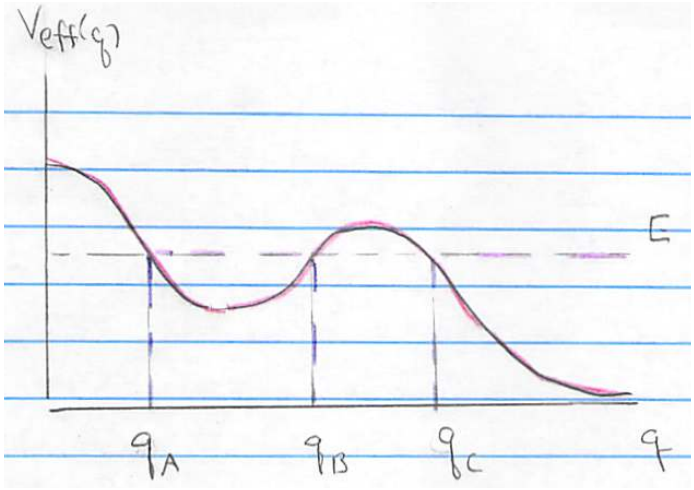
$$E = \frac{1}{2} m(q)\, \dot{q}^2 + V_{\text{eff}}(q) \tag{1.48}$$

You should be able to show that the this first integral equation can be used to determine $q(t)$ implicitly. Integrating from (t_0, q_0) to $(t, q(t))$ yields

$$\pm \int_{q_0}^{q(t)} dq \left(\frac{m(q)}{2(E - V_{\text{eff}}(q))} \right)^{1/2} = t - t_0\,, \tag{1.49}$$

which, when inverted, gives $q(t)$. The plus sign is when q is increasing in time, while the minus sign is when $q(t)$ is decreasing in time

- In a typical case the potential $V_{\text{eff}}(q)$ and energy E is shown below



For the specified energy, the motion is unbounded for $q > q_c$, and oscillates between when $q_A < q < q_B$. q_A, q_B and q_C are called turning points. The period $\mathcal{T}(E)$ is the time it takes to go from q_A to q_B and back. Thus half a period $\mathcal{T}(E)/2$ is the time it takes to go from q_A to q_B or

$$\frac{\mathcal{T}(E)}{2} = \int_{q_A}^{q_B} dq \left(\frac{m(q)}{2(E - V_{\text{eff}}(q))} \right)^{1/2} . \tag{1.50}$$

1.3 The Hamiltonian Formalism, the Routhian, and the Legendre Transform

The Hamiltonian formalism: basic version

- Let the Lagrangian be a convex function of the velocity $v_q \equiv \dot{q}$. In one dimension this means that the momentum $p = \partial L/\partial v_q$ is an increasing function of the velocity $v_q \equiv \dot{q}$, i.e $\partial^2 L/\partial \dot{q}^2 > 0$. This means there is one value of the velocity for given momentum p, $\dot{q}(p)$. Clearly $L \propto v^2$ is convex.

  In higher dimensions we require that $\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j$ is a positive definite matrix. This means that for a given value of p_i there is a unique value of the velocity vector $v_q^i \equiv \dot{q}^i(p)$ at fixed q.

- With convex function $L(\dot{q})$ a Legendre transform useful, and trades the velocity dependence of the Lagrangian dependence for the momentum dependence p of the Hamiltonian

  First note

$$dL = p\, d\dot{q} + \underbrace{\frac{\partial L}{\partial q} + \frac{\partial L}{\partial t} dt}_{\text{“spectators”}} \tag{1.51}$$

  We can trade the $d\dot{q}$ for dp by looking at $L - p\dot{q}$, or, as is conventional, minus this quantity. Thus we define

$$H(p, q, t) = p\, \dot{q}(p) - L(\dot{q}(p), q, t) \tag{1.52}$$

  where $\dot{q}(p)$ is determined from p at fixed q and t, i.e. we must invert the relation

$$p = \frac{\partial L(\dot{q}, q, t)}{\partial \dot{q}} \;\Rightarrow\; \text{determines } \dot{q}(p) \tag{1.53}$$

  We have (do it yourself!)

$$dH(p, q, t) = \dot{q}\, dp - (\underbrace{\frac{\partial L}{\partial q}\, dq + \frac{\partial L}{\partial t}\, dt}_{\text{“spectators”}}) . \tag{1.54}$$

  Thus we have

$$\frac{\partial H}{\partial p} = \dot{q} \qquad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{1.55}$$

  were L is a function $\dot{q}$ and H is a function of the corresponding p. You should be able to show that these results (together with the Euler-Lagrange equations) yield Hamilton's equations of motion:

$$\frac{dq}{dt} = \frac{\partial H(q, p, t)}{\partial p} \tag{1.56}$$

$$\frac{dp}{dt} = -\frac{\partial H(q, p, t)}{\partial q} \tag{1.57}$$

- When more variables are around then we simply sum over the $p_i \dot{q}^i$ term

$$H(p, q, t) = \sum_i p_i \dot{q}^i(p) - L(\dot{q}(p), q, t) \tag{1.58}$$

  and the equation of motion are

$$\frac{dq^i}{dt} = \frac{\partial H(q, p, t)}{\partial p_i} \tag{1.59}$$

$$\frac{dp_i}{dt} = -\frac{\partial H(q, p, t)}{\partial q^i} \tag{1.60}$$

- The total derivative of the Hamiltonian satisfies
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{1.61}$$
so that if H is not an explicitly function of time then it is constant.

- For a (rather general) Lagrangian of the form
$$L = \frac{1}{2}a_{ij}(q)\dot{q}^i\dot{q}^j + b_i(q)\dot{q}^i - U(q)\,, \tag{1.62}$$
the momenta and velocities are related via
$$p_i = a_{ij}\dot{q}^j + b_i\,, \qquad \dot{q}^i = (a^{-1})^{ij}(p_j - b_j)\,. \tag{1.63}$$
The Hamiltonian is
$$H(p,q,t) = \frac{1}{2}(a^{-1})^{ij}(p_i - b_i)(p_j - b_j) + U(q)\,. \tag{1.64}$$
This should be compared to the hamiltonian function in (1.46). The Hamiltonian is a function of the b_i, while the hamiltonian function is not. The Hamiltonian and hamiltonian function return the same value at corresponding points where $\dot{q} = \dot{q}(p)$, but have different functional forms.

The action principle

- The Hamiltonian can be used in the action principle to determine the equation of motion. The action takes a path in p, q space $(p_i(t), q^i(t))$ and returns a number
$$S[p(t), q(t), t] = \int \mathrm{d}t\, \left(p_i\,\dot{q}^i - H(p,q,t)\right) \tag{1.65}$$
We note $p_i\dot{q}^i - H = L$ at corresponding points. Varying the action with $p_i(t)$ and $q^i(t)$ separately (keeping the ends fixed) gives the Hamiltonian equation of motion. By doing this variation you should be able to show that
$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}\,, \tag{1.66}$$
$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}\,. \tag{1.67}$$

The Routhian

- It is often convenient to Legendre transform with respect to some of the coordinates. (This is usually convenient for the cyclic coordinates).

  Suppose we have two coordinates x and y, with Lagrangian $L(\dot{x}, x, \dot{y}, y)$. If we Legendre transform with respect to $\dot{x}$ (replacing it with p_x), but leave $\dot{y}$ alone:
$$R(p_x, x, \dot{y}, y) \equiv p_x\,\dot{x}(p_x) - L(\dot{x}(p_x), x, \dot{y}, y)\,, \tag{1.68}$$
then R (known as the Routhian) acts like a Hamiltonian for (p_x, x), but a Lagrangian[2] for $(\dot{y}, y)$. You should be able to show that
$$\frac{dx}{dt} = \frac{\partial R}{\partial p_x} \tag{1.69}$$
$$\frac{dp_x}{dt} = -\frac{\partial R}{\partial x} \tag{1.70}$$
$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{y}}\right) = \frac{\partial R}{\partial y} \tag{1.71}$$
Here, since the variables in R are $p_x, x, \dot{y}$ and y, the partial derivative, $\partial R/\partial y$, means, $(\partial R/\partial y)_{p_x}$. In the Lagrangian setup $L(\dot{x}, x, \dot{y}, y)$, with variables $\dot{x}, x, \dot{y}$ and y, one would have $(\partial L/\partial y)_{\dot{x}}$.

[2] Technically it is actually $-R$ that is Lagrangian for $\dot{y}, y$, due to the fact we are subtracting L when making the Legendre transform in Eq. (1.68). Of course you could have done the following $R = L - p_x\dot{x}$, and then it would be a Lagrangian for y, but $-R$ would be the Hamiltonian for x.

The Legendre Transform as extremization in the presence of an external bias (force)

- Consider the convex function $U(x)$. Its derivative is[3]

$$dU = f_0(x)\,dx \tag{1.72}$$

Then we define[4]

$$\hat{U}(x,f) = fx - U(x)\,. \tag{1.73}$$

Then the Legendre transform is the extremum (maximum or minimum) of $\hat{U}(x,f)$ for fixed f, i.e.

$$V(f) = \text{extrm}_x\ (f\,x - U(x))\ . \tag{1.74}$$

This means that we are to change x until we reach the value $x(f)$ where $\hat{U}$ is a maximum or minimum. The value of $\hat{U}$ at this point is $V(f)$. By differentiation, the extremal point is when $f = dU/dx = f_0(x)$, which must be inverted to determine $x(f)$. Then $V(f) = fx(f) - U(x(f))$.

- We have

$$dU = f(x)\,dx \qquad \text{and} \quad dV = x(f)\,df \tag{1.75}$$

and a relation between the second derivatives

$$\frac{d^2U}{dx^2}\frac{d^2V}{df^2} = 1 \tag{1.76}$$

- Then inverse Legendre transform returns the back the potential

$$U(x) = \text{extrm}_f\ (f\,x - V(f)) \tag{1.77}$$

which you should prove for yourself.

- For more degrees of freedom, take $U(x_1,x_2)$ for example, the procedure works similarly. We define

$$V(f_1,f_2) = \text{extrm}_{x_1,x_2}\left(f_1x^1 + f_2x^2 - U(x)\right) \tag{1.78}$$

Then

$$dU = f_1\,dx^1 + f_2\,dx^2 \qquad \text{and} dV = x^1\,df_1 + x^2\,df_2 \tag{1.79}$$

Note that the matrices of second derivatives

$$U_{ij} \equiv \frac{\partial^2 U}{\partial x^i \partial x^j} \qquad V^{ij} \equiv \frac{\partial^2 V}{\partial f_i \partial f_j} \tag{1.80}$$

are inverses of each

$$V^{i\ell}U_{\ell j} = \delta^i_j \tag{1.81}$$

[3]Think of $U(x)$ as the spring like potential that a particle feels. Then $f_0(x)$ is the external force that must be *applied* to the system so that the particle is in equilibrium at position x. The "internal" force that the potential gives is $f_{\text{internal}}(x) = -dU/dx$. This internal force must be counterbalanced by the applied force $f_0(x) = -f_{\text{internal}}(x)$.

[4]Referring to to the previous footnote $\hat{U}(x,f)$ is minus the potential in the presence of an applied external force f. In thermodynamics we would define the Legendre transform with $\hat{U} = U - fx$, but the overall sign leads only to minor differences. We follow the mechanics convention, $H = pv_q - L$, with regard to sign.

1.4 Motion in a Central Potential

Central potentials $U(r)$ and the Kepler Problem

- We have two bodies with m_1 and $\boldsymbol{r}_1$ and m_2 and $\boldsymbol{r}_2$, and generally take $\boldsymbol{r}_1$ to be the "earth" and $\boldsymbol{r}_2$ and sun. We first switch to center of mass $\boldsymbol{R}$ and relative coordinates $\boldsymbol{r}$

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{M}\,, \tag{1.82}$$

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2\,. \tag{1.83}$$

  with $M = m_1 + m_2$. We have the kinetic energy

$$T = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 \tag{1.84}$$

  where $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass, and thus the Lagrangian is

$$L = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - U(|\boldsymbol{r}|) \tag{1.85}$$

  where $U(|\boldsymbol{r}|)$ is the potential energy of the two particles.

- The overall center of mass motion does not change the orbital dynamics. We can choose $\boldsymbol{R} = \dot{\boldsymbol{R}} = 0$, so that the angular momentum of the center of mass is zero. Then the internal angular momentum is

$$\boldsymbol{L} = \mu\boldsymbol{r} \times \dot{\boldsymbol{r}} \tag{1.86}$$

  $\boldsymbol{L}$ can be chosen to lie along the z axis so that $\boldsymbol{r}$ lies in the x, y plane

$$\boldsymbol{r} = r\,(\cos\phi, \sin\phi, 0) \tag{1.87}$$

  The Lagrangian neglecting the center of mass motion is

$$L = \frac{1}{2}\mu(r^2 + r^2\dot{\phi}^2) - U(r) \tag{1.88}$$

- There are two integrals of motion for the motion in the effective potential:

$$\ell = \mu r^2\dot{\phi}\,, \tag{1.89}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r, \ell)\,. \tag{1.90}$$

  The effective particle with mass μ moves in the effective potential is

$$V_{\text{eff}}(r, \ell) = \frac{\ell^2}{2\mu r^2} + U(r)\,. \tag{1.91}$$

  Given the integrals of motion E and ℓ it is easy to determine $d\phi/d$ and dr/dt. From there it is straightforward to find an equation for $dr/d\phi = \dot{r}/\dot{\phi}$. Integrating $dr/d\phi$ gives the orbit for $r(\phi)$. This integral from (r_1, ϕ_1) to (r, ϕ) is

$$\phi - \phi_1 = \frac{\ell}{\sqrt{2\mu}}\int_{r_1}^{r} \frac{dr/r^2}{\sqrt{E - V_{\text{eff}}(r, \ell)}} \tag{1.92}$$

  for an arbitrary potential $U(r)$.

- For the coulomb potential $U = -k/r$, Eq. (1.92) for $r(\phi)$ can be integrated by making the "conformal" substitution

$$u \equiv \frac{1}{r} \qquad du = \frac{dr}{r^2}\,, \tag{1.93}$$

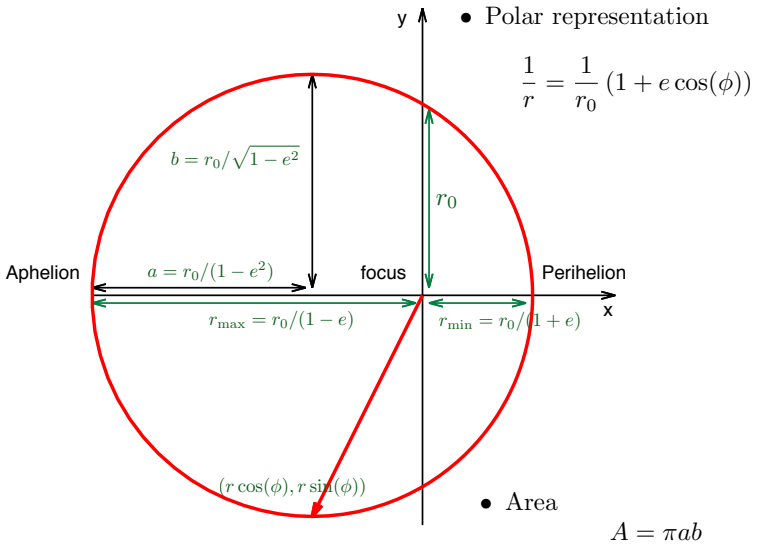


Figure 1.1:

leading to the equation of an ellipse:

$$\frac{1}{r} = \frac{1}{r_0}(1 + e\cos(\phi))\,. \tag{1.94}$$

r_0 is known as the lattice rectum (see figure for geometric meaning), and e is known as the eccentricity of the ellipse, which is a measure of how much the orbit deviates from a circle. A convenient summary of the elliptic geometry is given in Fig. 1.1

The parameters of the ellipse r_0 and e are determined by the integrals of motion, E and ℓ. The lattice rectum is determined by the angular momentum, $r_0 = \ell^2/\mu k$. The eccentricity e is determined by the excitation energy above the minimum of V_{eff} (with fixed ℓ). More explicitly $e = \sqrt{1 + E/\epsilon_0}$, with $\epsilon_0 = \ell^2/2\mu r_0^2$. When the energy of the orbit is at its minimum, $E = V_{\min} = -\epsilon_0$, then the eccentricity is zero and the radius is constant, i.e. the orbit is circular.

- The Coulomb potential has a characteristic scale $r_0 \sim \ell^2/\mu k$ when the potential k/r_0 and kinetic $\ell^2/\mu r_0^2$ are the same order of magnitude. Indeed, for a circular orbit of radius r_0, one shows by freshman physics that the radius is determined by the angular momentum, $r_0 = \ell^2/\mu k$. For such a circular orbits the kinetic energy is $\epsilon_0 \equiv \ell^2/2\mu r_0$ and is minus-half the potential $U = -k/r_0 = -2\epsilon_0$. The total energy (kinetic+potential) is $E = -\epsilon_0$ where

$$\epsilon_0 \equiv \frac{\ell^2}{2\mu r_0^2} = \frac{k}{2r_0}\,, \tag{1.95}$$

which explains the notation for the parameters in the previous item.

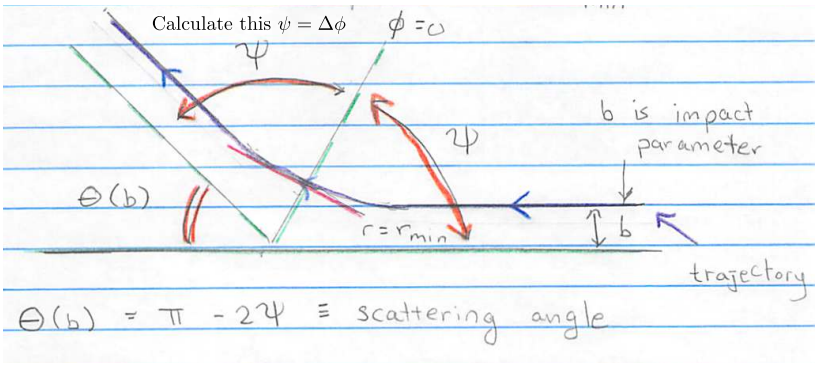
- For the Newton potential $U = -k/r$ and the spherical harmonic oscillator $U = \frac{1}{2}kr^2$ the orbits are closed (Bertrand's theorem). For no other central potentials are the orbits closed. The closed orbits are a consequence of an additional symmetry which we will discuss later.

Cross sections and scattering

- When considering the scattering problem we are interested in computing the scattering angle θ (the angle of deflection) for given energy E and impact parameter b. Here the impact parameter b is the transverse distance at large r from the target and is another way to record the angular momentum. At larger r the velocity is constant, $E = \frac{1}{2}mv^2$, and the angular momentum is

$$\ell = mvr\sin\theta = mvb = \sqrt{2mE}b \tag{1.96}$$

- The scattering angle $\theta(b)$ is shown below:



A particle comes in with impact parameters b (or angular momentum ℓ) and energy E, and is deflected by angle $\theta(b, E)$. From our mechanical perspective we find it easiest to compute the change in the angle ϕ as the particle propagates from its distance of closest approach $r_{\min}$ up to infinity. This is (the second) angle ψ in the figure above. It is related to $\theta(b, E)$ by simple geometry.

$$\theta(b) = \pi - 2\psi \,. \tag{1.97}$$

We have from Eq. (1.92)

$$\Delta\phi = \psi = \frac{\ell}{\sqrt{2m}} \int_{r_{\min}}^{\infty} \frac{dr/r^2}{(E - V_{\text{eff}}(r))^{1/2}} \,. \tag{1.98}$$

For the Coulomb problem $U = k/r$ this integration is straightforward with the substitution $u = 1/r$, and yields $\tan(\psi)$ and since $\psi = \pi/2 - \theta/2$

$$\cot(\theta/2) = \frac{2Eb}{k} \,. \tag{1.99}$$

- The scattering problem is usually phrased in terms of cross section:

  (i) Consider a beam of particles of luminosity $\mathscr{L}$. $\mathscr{L}$ is the number of particles crossing the target per area per time, and is also called the incident flux or intensity.

  (ii) The number of incoming particles which scatter per time $d\Gamma$ with impact parameter between b and db is $d\Gamma = \mathscr{L} 2\pi b|db|$. We put absolute values because we think of db as an positive interval.

  (iii) The number of incoming particles per time (or rate $d\Gamma$) which then end up at in ring of solid angle $d\Omega = 2\pi \sin(\theta)|d\theta|$ per time is

$$d\Gamma = \mathscr{L} \frac{b}{\sin\theta} \frac{|db|}{|d\theta|} \, d\Omega \,. \tag{1.100}$$

  So the scattering rate per solid angle is

$$\frac{d\Gamma}{d\Omega} = \mathscr{L} \frac{b}{\sin\theta} \frac{|db|}{|d\theta|} \,. \tag{1.101}$$

  The cross section is by definition the scattering rate divided by the incident flux

$$\frac{d\sigma}{d\Omega} \equiv \frac{1}{\mathscr{L}} \frac{d\Gamma}{d\Omega} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| \,. \tag{1.102}$$

  (iv) The cross section has units of area and gives a measure of the effective size of the target. It is usually measured in barns, $1\,\text{barn} = 10^{-24}\,\text{cm}^2$.

- For the Coulomb problem, we can different $d\theta/db$ (Eq. (1.99)) and use it in Eq. (1.102) to determine the Rutherford cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)} \sim \frac{1}{\theta^4}\,, \tag{1.103}$$

  which is inversely proportional to $1/\theta^4$ at small angles.

1.5 Constraints

Lagrange multipliers

- First we considered minimizing $U(x,y)$ subject to a constraint $Q(x,y)=0$. We said that we should instead minimize
$$\hat{U}(x,y,\lambda) = U(x,y) - \lambda Q(x,y)\,. \tag{1.104}$$
λ is known as a Lagrange multiplier[5]. This leads to the conditions
$$d\hat{U}(x,y) = \left(\frac{\partial U}{\partial x} - \lambda\frac{\partial Q}{\partial x}\right)dx + \left(\frac{\partial U}{\partial y} - \lambda\frac{\partial Q}{\partial y}\right)dy - Qd\lambda = 0 \tag{1.105}$$
where the terms in front of dx, dy, and $d\lambda$ should be set to zero. We explained that Q can be thought of as a generalized coordinate, and λ is a generalized force conjugate to Q. This is just like adding an external force. For instance if I have a potential $U(x,y)$ and add an external force f in the x direction then the new potential is
$$\hat{U}(x,y,f) = U(x,y) - fx\,. \tag{1.106}$$
The forces of constraint in the x and y directions are
$$F_x = \lambda\partial_x Q\,, \tag{1.107}$$
$$F_y = \lambda\partial_y Q\,. \tag{1.108}$$

- The setup easily generalizes to more coordinates and more constraints. For coordinates x^A and constraints $Q^\alpha(x^A)$ with $\alpha = 1\ldots m$, if we want to minimize $U(x^A)$ subject to these constraints, we instead extremize
$$\hat{U}(x^A) = U(x^A) - \lambda_\alpha Q^\alpha(x^A) \tag{1.109}$$
requiring that $d\hat{U}=0$, i.e. require
$$\frac{\partial\hat{U}}{\partial x^A} = 0 \tag{1.110}$$
$$\frac{\partial\hat{U}}{\partial\lambda_\alpha} = 0 \tag{1.111}$$
The forces of constraint in the x^A direction are
$$F_A = \lambda_\alpha\frac{\partial Q^\alpha}{\partial x^A} \tag{1.112}$$

Newton's Laws and Lagrange with constraints

- Consider Newton's Laws for particles with positions $\boldsymbol{r}_a$. For simplicity consider just one constraint.
$$Q(\boldsymbol{r}_a) = 0 \tag{1.113}$$
Then
$$dQ = \nabla_{\boldsymbol{r}_a} Q\cdot d\boldsymbol{r}_a = 0 \tag{1.114}$$
The forces of constraints$\boldsymbol{F}_a^C$ do no work
$$\boldsymbol{F}_a^C\cdot d\boldsymbol{r}^a = 0 \tag{1.115}$$

[5]The sign in front of λ is irrelevant. The choice here is so that λ corresponds to the generalized force in the direction of increasing Q, compare to Eq. (1.106). When we consider contraints in the Lagrangian, $L=T-U$, the multipliers will then come with a plus sign $\hat{L}=T-U+\lambda Q$.

Thus, we make take $\boldsymbol{F}_a^C$ to be proportional to the gradient of Q

$$\boldsymbol{F}_a^C = \lambda \nabla_{\boldsymbol{r}_a} Q \tag{1.116}$$

Then Newton's Laws read

$$\frac{d\boldsymbol{p}_a}{dt} = \boldsymbol{F}_a^{\text{ext}} + \lambda \nabla_{\boldsymbol{r}_a} Q\,. \tag{1.117}$$

Then Newton's Law ($\boldsymbol{F} = m\boldsymbol{a}$) *and* the constraint, determine the accelerations of the particles *and* the magnitude of the forces of constraint, i.e. λ.

- You should do some simple problems on Attwood's machines (see below) to convice yourself that we are always solving Eq. (1.117) when doing Freshmann physics problems.

- In the Lagrangian formalism we add some lagrange multipliers to enforce the constraints. Instead of extremizing $L(\dot{\boldsymbol{r}}_a, \boldsymbol{r}_a)$, one extremizes $\hat{L}(\dot{\boldsymbol{r}}_a, \boldsymbol{r}_a, \lambda) = L + \lambda Q$, where λ is like an extra coordinate. The Euler-Lagrange equations for $\hat{L}$ are[6]

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{\boldsymbol{r}}_a}\right) = \frac{\partial \hat{L}}{\partial \boldsymbol{r}_a} \tag{1.120}$$

$$0 = Q \tag{1.121}$$

- If there are more constraints Q^α, simply make the replacement $\lambda Q \to \lambda_\alpha Q^\alpha$ in the lagrangian formalism. In the Newtonian formalism the force of constraint on the a-th particle is

$$\boldsymbol{F}_a = \lambda_\alpha \nabla_{\boldsymbol{r}_a} Q^\alpha\,. \tag{1.122}$$

- *Attwood machine*. Consider two masses m_1 and m_2 hanging over a massless pulley (you know the problem!). We have two coordinates z_1 and z_2 where z_1 and z_2 are the distances below the pulley (increasing z means further down). The constraint is

$$Q = z_1 + z_2 - L \tag{1.123}$$

The hatted Lagrangian is

$$\hat{L} = \tfrac{1}{2} m_1 \dot{z}_1^2 + \tfrac{1}{2} m_2 \dot{z}_2^2 + m_1 g z_1 + m_2 g g z_2 + \lambda(z_1 + z_2 - L) \tag{1.124}$$

Newton's or Lagranges' equation of motion are

$$m_1 a_1 = m_1 g + \lambda \tag{1.125}$$

$$m_2 a_2 = m_2 g + \lambda \tag{1.126}$$

$$z_1 + z_2 = L \tag{1.127}$$

Which are easily solved for a_1, a_2 and λ, using that Eq. (1.127) implies by differentationg that $a_1 + a_2 = 0$. Solving these equations gives λ negative, i.e. the force is up not down. The case when the pulley has mass in the Lagrangian formalism is suggested as an excercise.

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[6] Perhaps we should write it a bit more explicitly. The coordinates of $\boldsymbol{r}_a$ are r_a^i with $i = x, y, z$. We mean

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{r}_a^i}\right) = \frac{\partial \hat{L}}{\partial r_a^i} \tag{1.118}$$

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{\lambda}}\right) = \frac{\partial \hat{L}}{\partial \lambda} \tag{1.119}$$

The equation $0 = Q$ follows from the equation for λ, which simply enforces the constraint.