Physics 501: Classical Mechanics

Midterm Exam

Stony Brook University

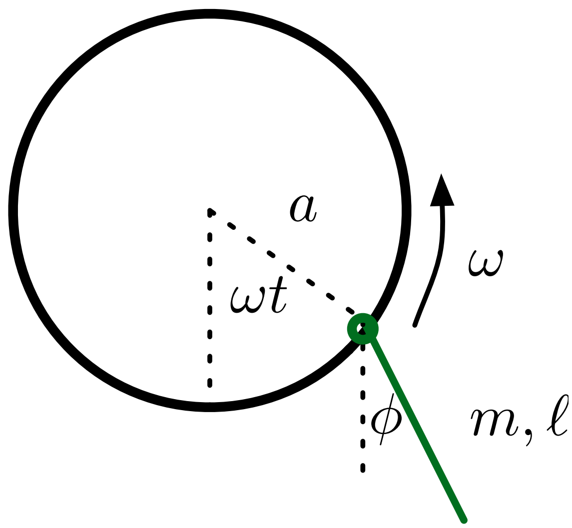
Fall 2020

General Instructions:

You may use one page (front and back) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Problem 1. A pendulum on a wheel

Consider a pendulum consisting of a uniform rod of length ℓ and mass m in the earth's gravitational field. The pivot point of the pendulum is attached to the rim of a wheel of radius a which turns with angular velocity ω, but otherwise the angle of the pendulum is able to rotate freely around its pivot point.



(a) Determine the Lagrangian of the system. You should find after suitable manipulations that the Lagrangian can be written

$$L = \frac{1}{6} m\ell^2 \dot{\phi}^2 + \frac{1}{2} ma\ell\omega^2 \cos(\omega t - \phi) - \frac{1}{2} mg\ell \cos\phi \tag{1}$$

You will be graded on the clarity of the derivation not the answer.

(b) Describe the motion qualitatively when ω is small, and when ω is large? Define what is meant by fast and slow in this context. When ω is arbitrarily fast determine the determine the steady state value of ϕ as a function of time.

(c) When ω is fast, but not arbitrarily fast, the steady state of part (b) will be perturbed by gravity, and ϕ will oscillate around its steady state value. Expand the Lagrangian to quadratic order in $\delta\phi$, and find the resulting equation of motion for $\delta\phi$.

(d) Determine the steady state amplitude and frequency of the resulting oscillations to lowest non-trivial order in the gravitational perturbation, assuming that you are far from any resonance.

(e) If the rotational frequency is twice the resonant frequency of the oscillator of part (c), the perturbative expansion developed in (d) will break down. Explain why.

Problem 2. General variation

Consider the action

$$S[q(t); q_1, t_1, q_2 t_2] = \int_{t_1}^{t_2} dt L(q, \dot{q}, t) \tag{2}$$

evaluated on a trajectory $q(t)$ with $t \in [t_1, t_2]$ with endpoints $q(t_1) = q_1$ and $q(t_2) = q_2$. Determine how the action is changed by a general variation

$$\begin{aligned} q(t) \to& q + \delta q(t) &(3)\\ t_1 \to& t_1 + \delta t_1 &(4)\\ t_2 \to& t_2 + \delta t_2 &(5) \end{aligned}$$

Assume that $q(t)$ satisfies the equations of motion, but do not assume that $\delta q(t)$ vanishes at the end points. Explain all of your steps.

Problem 3. A curious Lagrangian

Consider a Lagrangian

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega_0^2 q^2 + \frac{\kappa}{4}\dot{q}^4 \tag{6}$$

(a) The motion is started with initial conditions

$$q(0) = x_0 \qquad \dot{q}(0) = 0 \tag{7}$$

Under what conditions can the term $\kappa\dot{q}^4/4$ be considered a small perturbation?

(b) Determine the motion system treating $\kappa\dot{q}^4/4$ as perturbation, and working to 1st order in secular perturbation theory.