Physics 501: Classical Mechanics

Final Exam

Stony Brook University

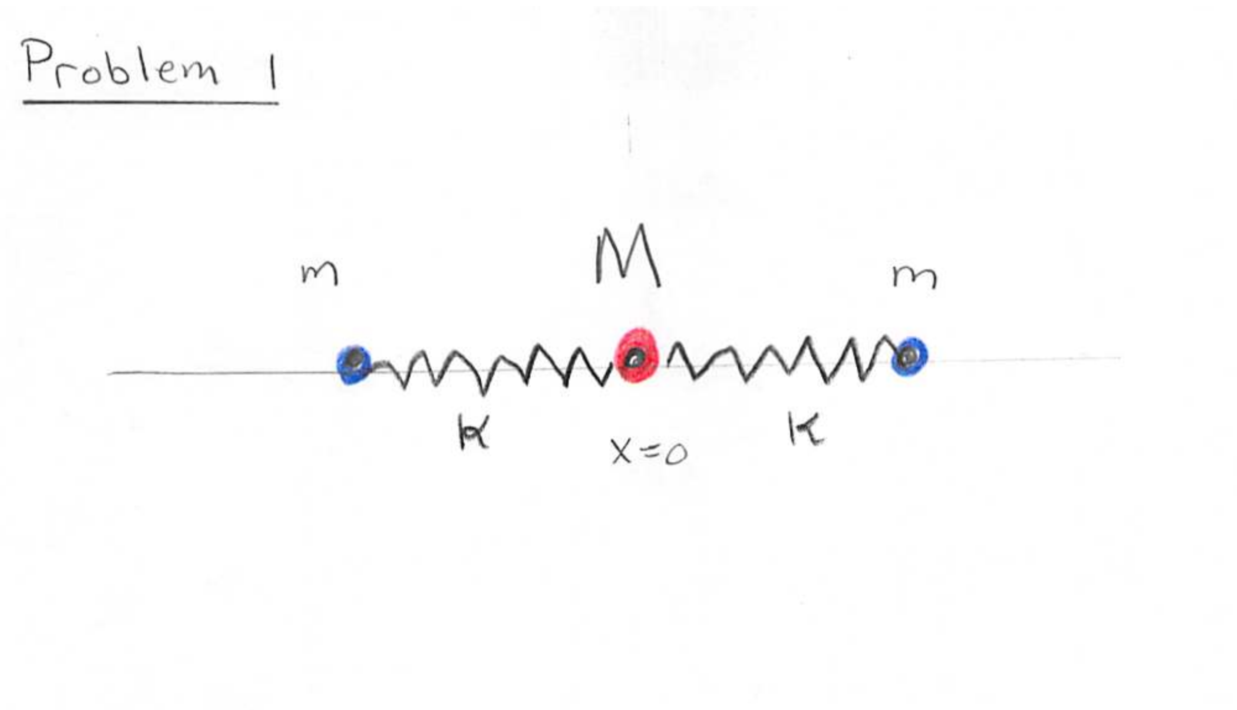
Fall 2018

General Instructions:

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

1 A kicked oscillator

Consider three particles of masses m, M, m, constrained to move along the x-axis. The particles are connected by ideal unstretched springs with spring constant κ as shown below (see the last page).



Neglect friction. The unstretched distance between the particles is a.

(a) Determine the normal modes of the system, and make a sketch of them. Identify any zero modes.

(b) At time $t = 0$, the central particle (with mass M) is given an impulsive kick $F_0 = P_0\delta(t)$ to the right. Determine the the position of each particle as a function of time for $t > 0$.

Assume that the impulse is small enough that subsequent oscillations can be treated in a harmonic approximation.

Solution:

(a) We first write down the Lagrangian

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}M\dot{X}^2 - \frac{1}{2}\kappa(x_1 - x_2)^2 - \frac{1}{2}\kappa(x_2 - X)^2 \tag{1}$$

The equation of motion is

$$m\ddot{x}_1 = -\,\kappa(x_1 - x_2) \tag{2}$$
$$M\ddot{X} = \kappa(x_1 - X) - \kappa(x_2 - X) \tag{3}$$
$$m\ddot{x}_2 = +\,\kappa(x_2 - X) \tag{4}$$

which in matrix form reads

$$\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{X} \\ \ddot{x}_2 \end{pmatrix} = \kappa \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} \tag{5}$$

The most direct way is to find the normal modes is substitute a form

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = e^{-i\omega t} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{6}$$

leading to

$$\begin{pmatrix} \kappa - m\omega^2 & -\kappa & 0 \\ -\kappa & 2\kappa - M\omega^2 & -\kappa \\ 0 & -\kappa & \kappa - m\omega^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0\,. \tag{7}$$

Then these linear equations will have non-trivial solutions when the determinant vanishes

$$\det \begin{pmatrix} \kappa - m\omega^2 & -\kappa & 0 \\ -\kappa & 2\kappa - M\omega^2 & -\kappa \\ 0 & -\kappa & \kappa - m\omega^2 \end{pmatrix} = 0 \tag{8}$$

Leading to

$$-\omega^2(\kappa - m\omega^2)(\kappa(2m + M) - mM\omega^2) = 0 \tag{9}$$

So

$$\omega = 0 \qquad \omega^2 = \frac{\kappa}{m} \qquad \omega^2 = \frac{\kappa}{m}(2m/M + 1) \tag{10}$$

Here we will follow a slightly less systematic approach based on symmetry. First of all there is an obvious zero mode corresponding to a shift of all particles by a finite amount. The potential energy (which depends only on differences) is obviously unchanged by this shift. The zero mode is

$$\boxed{\vec{E}_z = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \omega^2 = 0} \tag{11}$$

Thus a specific solution to the EOM for belonging only zero mode is

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = (A_z t + B_z) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{12}$$

Then we can guess the next two modes based on the symmetry of the problem. The modes should be even (see figure below) and odd (see figure below) under the reflection symmetry of the problem. The even mode is

$$\boxed{\vec{E}_e = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} \tag{13}$$

and is clearly orthogonal to the zero mode. This means, more physically, that this oscillation does not displace the center of mass. The odd mode takes the form

$$\boxed{\vec{E}_o = \begin{pmatrix} 1 \\ \alpha \\ 1 \end{pmatrix}} \tag{14}$$

We require that it be orthogonal to the zero mode,

$$\vec{E}_o^T \cdot \mathcal{M} \cdot \vec{E}_z = m + M\alpha + m = 0\,, \tag{15}$$

or equivalently we the odd mode should not displace the center of mass. Thus

$$\boxed{\alpha = -\frac{2m}{M}} \tag{16}$$

Now we can substitute these two modes into the the equation of motion and determine the frequencies. First we study the even mode, substituting this form into the equations of motion, taking

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = A_e e^{-i\omega_e t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\,, \tag{17}$$

we find

$$-m\omega_e^2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \kappa \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{18}$$

which satisfies the provided

$$\boxed{\omega_e^2 = \frac{\kappa}{m}} \tag{19}$$

Similarly we substitute

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = A_o e^{-i\omega_e t} \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}\,, \tag{20}$$

into the EOM (Eq. (5)), leading to

$$m\omega_o^2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = k(1 + 2m/M) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{21}$$

Thus the EOM are satisfied provided

$$\boxed{\omega_o^2 = \frac{k}{m}\left(1 + 2m/M\right)} \tag{22}$$

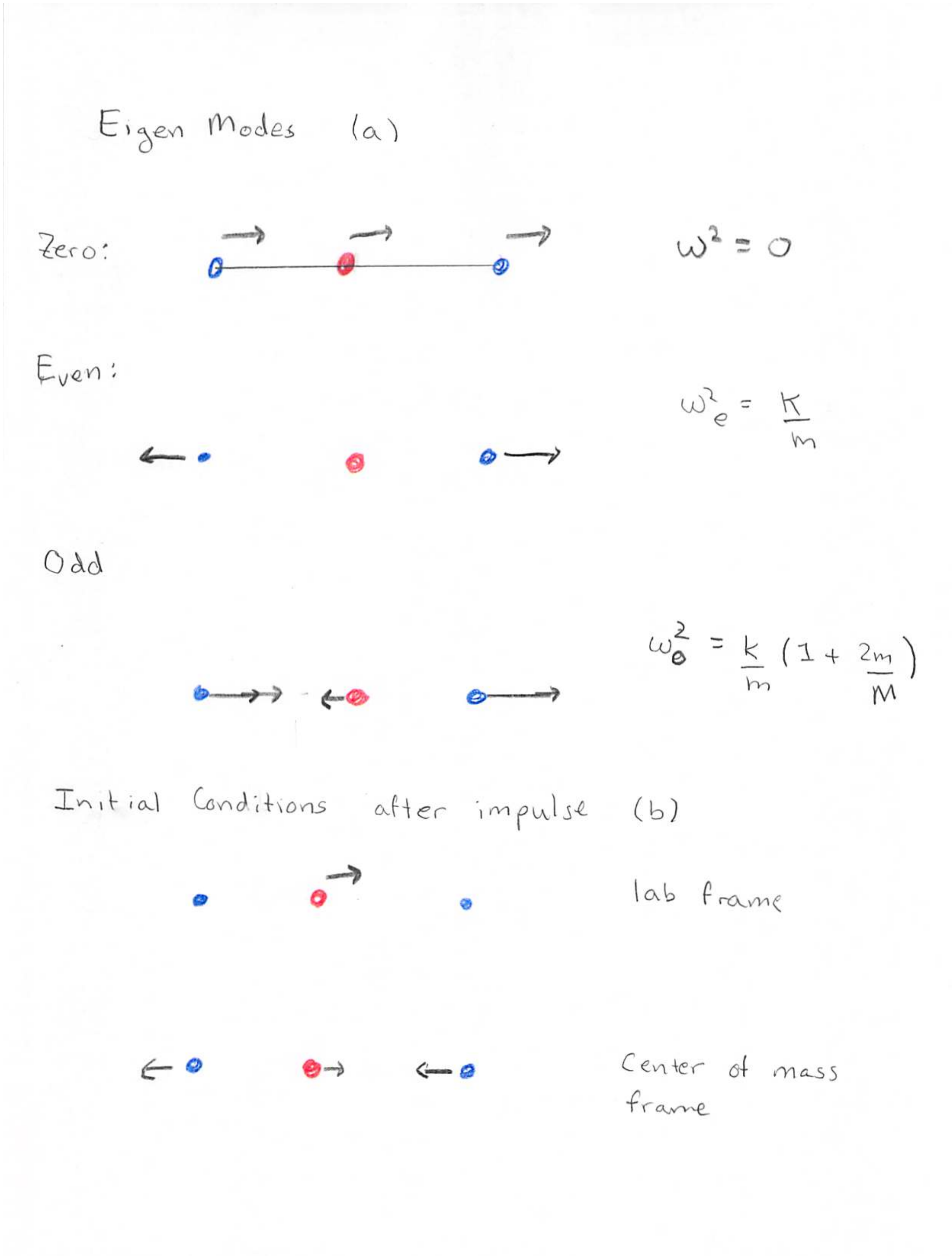


Figure 1: (a) A sketch of the normal modes. (b) The initial conditions after the impulse

(b) Here we need to analyze the motion just after the impulse. Just after the impulse, on the middle particle has any net velocity. Indeed the velocity of the middle particle just after the impulse is simply P_0/M and the velocity of the center of mass is $P_0/(M+2m)$, and the positions of the particles are unchanged (see figur!). Thus the initial conditions for our mechanics problem are

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_X \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ P_0/M \\ 0 \end{pmatrix} \tag{23}$$

The general solution is

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = (A_z t + B_z)\vec{E}_z + A_e \cos(\omega_e t + \varphi_e)\vec{E}_e + A_o \cos(\omega_o t + \varphi_o)\vec{E}_o \tag{24}$$

In general we should adjust the integration constants to match the initial conditions. Straightforward algebra can determine these constants.

To carry out this algebra it helps to have intuition. First, since no particle is displaced at $t = 0_+$ the solution must take the form

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = A_z t\vec{E}_z + A_e \sin(\omega_e t)\vec{E}_e + A_o \sin(\omega_o t)\vec{E}_o \tag{25}$$

This already simplifies the algebra. It is simple enough to stop thinking here and to solve for A_z, A_e, A_o in a pedestrian fashion.

But to clarify the physics we note that after the impulse, the center of mass moves with velocity $v_{cm} = P_0/(M+2m)$. This center of mass motion determines the zero mode, fixing $A_z = v_{\rm cm}$, and $B_z = 0$. Just after the impulse the mass M is moving with speed $v_0 = P_0/M$. In the center of mass frame (see figure) the particles are moving (just after the impulse) with velocity

$$\begin{pmatrix} -v_{\rm cm} \\ v_0 - v_{\rm cm} \\ -v_{\rm cm} \end{pmatrix} \tag{26}$$

which clearly belongs to the odd mode. We may therefore the set the amplitude of the even mode to zero. Then finally we may fix the amplitude of the odd from our initial conditions in the center of mass frame:

$$\begin{pmatrix} -v_{\rm cm} \\ v_0 - v_{\rm cm} \\ -v_{\rm cm} \end{pmatrix} = \frac{d}{dt}(A_0 \sin(\omega_o t))\Big|_{t=0_+} \begin{pmatrix} 1 \\ \frac{-2m}{M} \\ 1 \end{pmatrix} . \tag{27}$$

So we require $-A_o\omega_o = v_{\rm cm}$ leading to our final result

$$\boxed{\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = \frac{P_0 t}{M+2m}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{P_0}{(M+2m)\omega_0}\sin(\omega_o t)\begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix} .} \tag{28}$$

2 Hamiltonian basics

(a) Consider an infinitesimal canonical transformation generated by $G(q,p)$ for a Hamiltonian system with one generalized coordiante q. Show that if the transformation is a symmetry of a time independent Hamiltonian then $G(q,p)$ is constant in time.

(b) Consider a Hamiltonian system with Hamiltonian $H(q,p)$ and one generalzied coordinate q. Show that the volume of phase space is preserved under the time evolution of the system.

Solution:

(a) The transformation is

$$q \to q' = q + \lambda \frac{\partial G}{\partial p}\,, \tag{29}$$

$$p \to p' = p - \lambda \frac{\partial G}{\partial q}\,. \tag{30}$$

If this is a symmetry then the Hamiltonian is unchanged by the change of variables

$$H(q',p') = H(q,p)\,. \tag{31}$$

Substituting q' and p', and expanding to first order in λ

$$H(q',p') = H(q + \lambda \frac{\partial G}{\partial p}, p - \lambda \frac{\partial G}{\partial q}) \tag{32}$$

$$= H(q,p) + \lambda \underbrace{\left(\frac{\partial H}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial G}{\partial q} \right)}_{\equiv \{H,G\}} \tag{33}$$

Thus we find

$$\{H,G\} = 0\,. \tag{34}$$

Since the equation of motion of G is

$$\dot{G} = -\{H,G\}\,, \tag{35}$$

we find $\dot{G} = 0$, i.e. G is constant in time.

(b) We should look at how the measure

$$dq'\,dp' \tag{36}$$

transforms under time evolution. Under a small time step δt q,p are mapped to q',p' by the Hamiltonian evolution

$$q \to q' = q + \frac{\partial H}{\partial p}\delta t\,, \tag{37}$$

$$p \to p' = p - \frac{\partial H}{\partial q}\delta t\,. \tag{38}$$

So under the transformation the $dq'dp'$ is related to $dqdp$ by the Jacobian determinant:

$$dq'dp' = \frac{\partial(q', p')}{\partial(q, p)} dqdp \equiv \begin{vmatrix} \frac{\partial q'}{\partial q} & \frac{\partial q'}{\partial p} \\ \frac{\partial p'}{\partial q} & \frac{\partial p'}{\partial p} \end{vmatrix} dq\,dp \tag{39}$$

We may compute this determinant to first order in δt

$$dq'dp' = \begin{vmatrix} 1 + \frac{\partial^2 H}{\partial q \partial p}\delta t & \frac{\partial^2 H}{\partial q \partial p}\delta t \\ -\frac{\partial H}{\partial q \partial p}\delta t & 1 - \frac{\partial H}{\partial p \partial q}\delta t \end{vmatrix} dq\,dp \tag{40}$$

$$= \left(1 + \left(\frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial q \partial p}\right)\delta t\right) dqdp + O(\delta t^2) \tag{41}$$

$$= dp\,dp \tag{42}$$

which shows that the volume is conserved as the system evolves.

3 A damped anharmonic oscillator

Consider a damped anharmonic oscillator

$$m\frac{d^2x}{dt} + m\eta\frac{dx}{dt} + m\omega_0^2 x + \lambda x^3 = 0 \tag{43}$$

(a) At time $t = 0$ the oscillator is initialized with initial amplitude, $x(0) = A$, and no initial velocity. (i) Introduce an appropriate set of dimensionless variables to rewrite Eq. (43) in dimensionless form. (ii) For what range of the dimensionful parameters of this problem can the motion of the oscillator be consider approximately harmonic, with small damping and small anharmonic terms?

(b) Determine a zeroth order solution to Eq. (43), which is approximately valid for all times $t > 0$ for small damping and small anharmonic terms.

The identity $(\cos(x))^3 = \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x)$ may be useful.

Solution:

(a) We first set $\omega_0 = A = 1$. Then

$$\hat{x} = \frac{x}{A} \qquad \hat{t} = \omega_0 t \tag{44}$$

The equation then reads

$$m\omega_0^2 A \frac{d^2\hat{x}}{d\hat{t}^2} + (m\omega_0^2 A)\frac{\eta}{\omega_0}\frac{d\hat{x}}{d\hat{t}} + m\omega_0^2 A\hat{x} + \lambda A^3\hat{x}^3 = 0 \tag{45}$$

After dividing by $m\omega_0^2 A$ we find

$$\frac{d^2\hat{x}}{d\hat{t}^2} + \frac{\eta}{\omega_0}\frac{d\hat{x}}{d\hat{t}} + \hat{x} + \frac{\lambda A^2}{m\omega_0^2}\hat{x}^3 = 0 \tag{46}$$

So we require that

$$\frac{\eta}{\omega_0} \ll 1 \tag{47}$$

and

$$\frac{\lambda A^2}{m\omega_0^2} \ll 1 \tag{48}$$

Finally we define

$$\hat{\eta} = \frac{\eta}{\omega_0} \qquad \hat{\lambda} \equiv \frac{\lambda A^2}{m\omega_0^2} \tag{49}$$

and we will drop the "hats" below in all expressions.

(b) We substitute a trial solution and define the phase Ω

$$x(t) = C(t)\cos(-t + \varphi(t)) \qquad \Omega \equiv -t + \varphi \tag{50}$$

The spirit here is that in the absence of the damping and anharmonic terms the general solution is

$$x(t) = C\cos(-t + \varphi)\,. \tag{51}$$

In secular perturbation theory the integration constants C, φ are promoted to slow functions of time. This approximation scheme goes by various names such as the "slow-roll" approximation, or the "rotating waver approximation".

Computing the derivative

$$\frac{dx}{dt} = -\,C\sin(-t+\varphi)(-t+\dot{\varphi}) + \dot{C}\cos\Omega \tag{52}$$

$$= C\sin(\Omega) - C\sin(\Omega)\dot{\varphi} + \dot{C}\cos\Omega \tag{53}$$

Differentiating again

$$\frac{d^2x}{dt^2} = C\cos(\Omega)(-1+\dot{\varphi}) + \dot{C}\sin\Omega + C\cos(\Omega)\dot{\varphi} + \dot{C}\sin\Omega + \text{small} \tag{54}$$

$$= -\,C\cos\Omega + 2\dot{C}\sin\Omega + 2C\cos\Omega\dot{\varphi} \tag{55}$$

In this step we have neglected terms of order $\ddot{C}$ and $\ddot{\varphi}$. This is the spirit of the "slow-roll" approximation where the integation "constants" depend slowly on time.

Thus

$$\ddot{x} + x = 2\dot{C}\sin\Omega + 2C\cos\Omega\dot{\varphi} \tag{56}$$

The anharmonic term apprixmates to

$$\lambda x^3 = \frac{3}{4}\cos(\Omega) + \frac{1}{4}\cos(3\Omega) \tag{57}$$

$$\simeq \frac{3}{4}\cos(\Omega) \tag{58}$$

while the dissipative term is

$$\eta\frac{dx}{dt} = \eta C\sin(\Omega) \tag{59}$$

In these terms which are small because of the small parametes, λ and η, we may neglect $\dot{C}$ and $\dot{\varphi}$. The 3Ω terms contributes to the first correction, but do not contribute to the leading order solution (see lecture). It will be neglected below.

Collecting the partial results for the second derivate, Eq. (56), the anharmonic equation, Eq. (58), and the dissipative equation, Eq. (59), the equation of motion then reads

$$\left(2\dot{C} + \eta C\right)\sin\Omega + \left(2C\dot{\varphi} + \frac{3}{4}\lambda C^3\right) = 0 \tag{60}$$

Integrating the first equation

$$C(t) = C(0)e^{-(\eta/2)t} \tag{61}$$

At early times our solution must behave as $\cos(-kt)$, where k is a constant to be determined. This is a consequence of our initial conditions, where the initial amplitude is unity and the initial velocity is zero. We thus set $C(0)$ to unity.

Then we integrate the phase

$$\dot{\varphi} = \frac{3\lambda}{8}C^2 = \frac{3}{8}\lambda e^{-\eta t}\,, \tag{62}$$

Integrating the phase we find

$$\varphi = \frac{3\lambda}{8\eta}(1 - e^{-\eta t})\,, \tag{63}$$

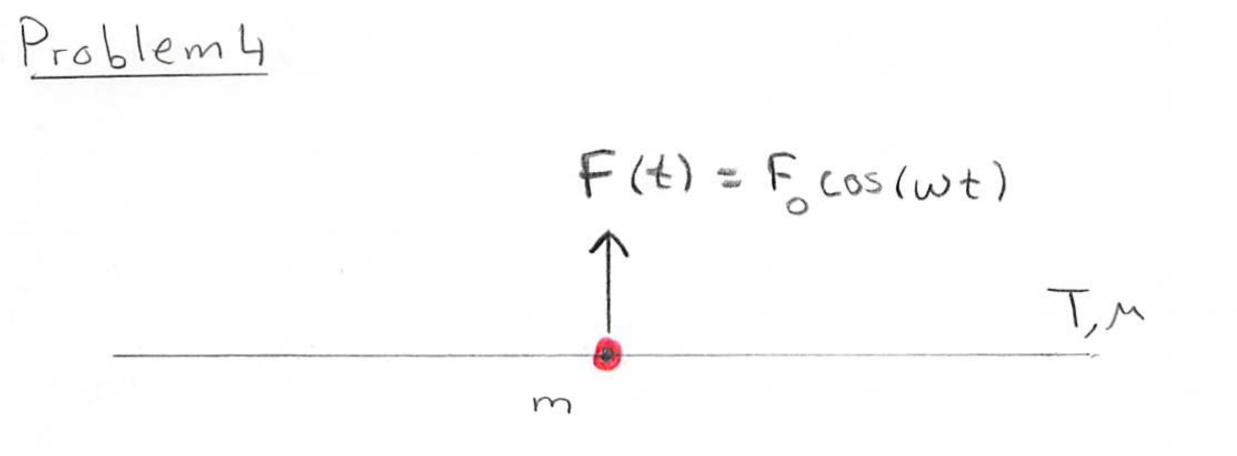
In integrating the phase we have specified that at small times the phase is $\varphi \propto t$ as required by our initial conditions that the initial velocity be zero. Summarizing then we find

$$\boxed{x(t) = e^{-(\eta/2)t}\cos\left[-t + \frac{3\lambda}{8\eta}(1 - e^{-\eta t})\right]\,.} \tag{64}$$

4 Oscillations on a string

Consider a long string of mass density μ and tension T running along the x-axis. Attached to the center of the string is a mass, m, as shown below (see the last page). The mass is pushed and pulled by an an external harmonic force, $F(t) = F_0 \cos(\omega t) = \mathrm{Re}[F_0 e^{-i\omega t}]$, direction.

The amplitude of the strings motion $y(t, x)$ is small enough to be treated in a harmonic approximation.



(a) Determine the steady state solution, $y(t, x)$, for the amplitude of the string to the right and left of the mass.

(b) Compute the time averaged work done per time by the external force.

Solution:

(a) The first part we recognize that the solution to right and the left must take the forms

$$y_L = Ae^{-ikx-i\omega t} \tag{65}$$
$$y_R = Be^{ikx-i\omega t} \tag{66}$$

Specifically the left solution consists only of left-movers, while the right solution consists only of right movers. Continuity demands that we set $A = B$. Drawing a free body digram of the mass (see below) which moves as $y_m(t)$ we determine a final constraint:

$$m\frac{d^2y_m}{dt^2} = F(t) + T\frac{dy_R}{dx} - T\frac{dy_L}{dx} \tag{67}$$

The mass moves up and down harmonically. Continuity says

$$y_m(t) = Ae^{-i\omega t} \tag{68}$$

Substituting the above forms into we find

$$-m\omega^2 A = F_0 + TikA - T(-ikA) \tag{69}$$

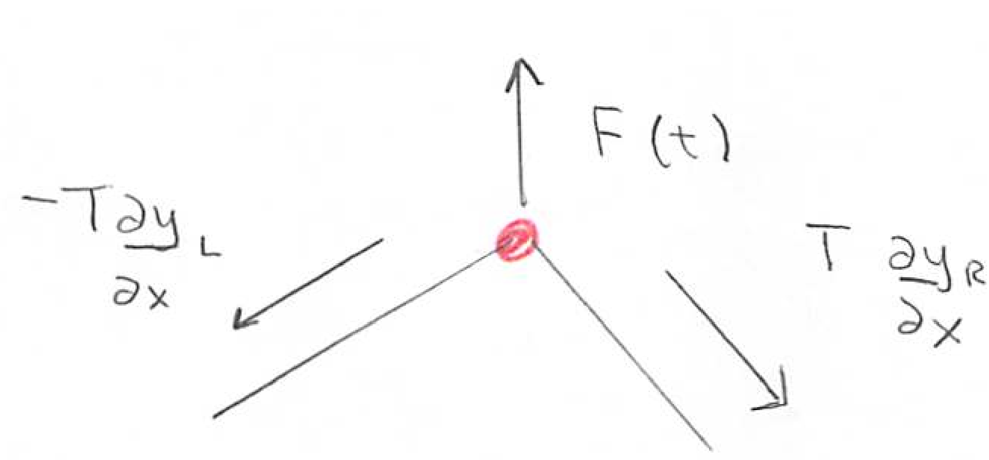


Figure 2: A free body diagram for the motion of the mass m.

So since $Tik = iZ\omega$ with $Z = \sqrt{T\mu}$ the wave impedance we find

$$A = \frac{F_0}{-m\omega^2 + 2iZ\omega} \tag{70}$$

This completes the solution

$$\boxed{y(t,x) = \begin{cases} \frac{F_0}{-m\omega^2+2iZ\omega} e^{-ikx-i\omega t} & x < 0 \\ \frac{F_0}{-m\omega^2+2iZ\omega} e^{ikx-i\omega t} & x > 0 \end{cases}} \tag{71}$$

Taking the real part this can be written

$$\boxed{y(t,x) = \frac{F_0}{\sqrt{(m\omega)^2 + (2Z\omega)^2}} \cos(k|x| - vt + \phi_0) \qquad \phi_0 = \tan^{-1}(2Z\omega/m\omega^2)} \tag{72}$$

(b) The work done is

$$\frac{dW}{dt} = \dot{y}_m F(t) \tag{73}$$

Since these quantities vary harmonically as

$$y_m = Ae^{-i\omega t} \qquad \dot{y}_m = -i\omega A e^{-i\omega t} \qquad F(t) = F_0 e^{-i\omega t} \tag{74}$$

we can compute the average

$$\overline{\frac{dW}{dt}} = \frac{1}{2}\mathrm{Re}[(-i\omega A)\, F_0^*] \tag{75}$$

Substituting the amplitude from Eq. (70) into this last expression we find

$$\boxed{\overline{\frac{dW}{dt}} = \frac{1}{2}\mathrm{Re}\left[\frac{-i\omega F_0}{-m\omega^2 + 2iZ\omega} F_0^*\right] = \frac{|F_0|^2\omega^2 Z}{(m\omega^2)^2 + (2Z\omega)^2}} \tag{76}$$