Physics 501: Classical Mechanics

Final Exam

Stony Brook University

Fall 2019

General Instructions:

You may use one page (front and back) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

1 Double pendulums

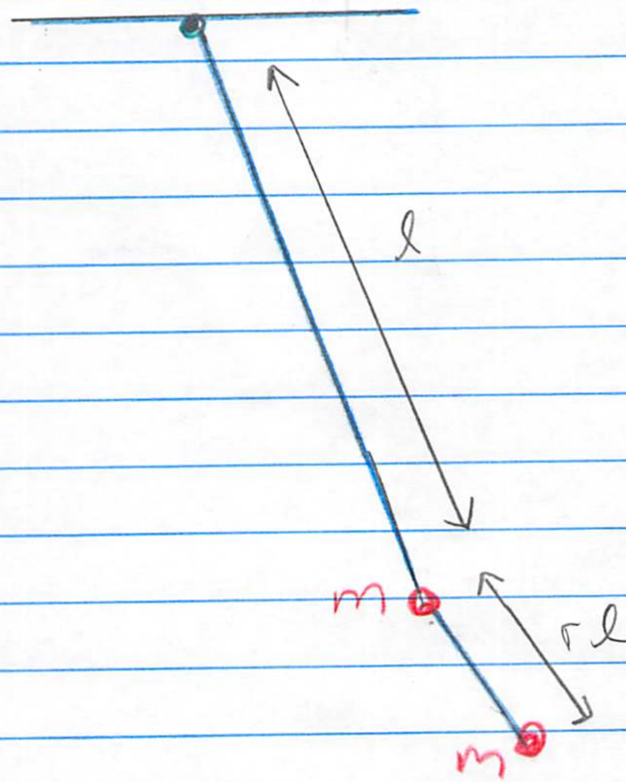
Consider two light rods coupled together with a hinge. The length of the first rod is ℓ, and it has a mass m at its end (see figure). The length of the second rod is a factor r of smaller than the first (i.e. its length is $r\ell$), and it also has mass m on its end. The system oscillates harmonically in the earth's gravitational field (see figure).

(a) Write down the Lagrangian of the system in a harmonic approximation. Check your work.

(b) Determine frequencies of the normal modes.

(c) Describe physically what is going on in the limit when $r \ll 1$.

Double Pendulums



Solution

(a) Writing the coordinates with small angle approximations $\sin\theta = \theta$, and $\cos\theta = 1 - \theta^2/2$ we have

$$x_1 = \ell\theta_1 \tag{1}$$
$$y_1 = -\ell + \ell\frac{\theta_1^2}{2} \tag{2}$$
$$x_2 = \ell\theta_1 + r\ell\theta_2 \tag{3}$$
$$y_2 = -\ell + \ell\frac{\theta_1^2}{2} + -r\ell + r\ell\frac{\theta_1^2}{2} \tag{4}$$

We find

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{y}_2^2 - mgy_1 - mgy_2 \tag{5}$$

So up to constants and keeping only harmonic terms

$$L = \frac{1}{2}m\ell^2\dot{\theta}_1^2 + \frac{1}{2}m\ell^2(\dot{\theta}_1 + r\dot{\theta}_2)^2 - \frac{1}{2}mg\ell\theta_1^2 - \frac{1}{2}mg\ell(\theta_1^2 + r\theta_2^2) \tag{6}$$

Regrouping terms

$$L = \frac{1}{2}m\ell^2\left(2\dot{\theta}_1^2 + 2r\dot{\theta}_1\dot{\theta}_2 + r^2\dot{\theta}_2^2\right) - \frac{1}{2}mg\ell\left(2\theta_1^2 + r\theta_2^2\right) \tag{7}$$

(b) So the equation of motion takes the form

$$\begin{pmatrix} 2 & r \\ r & r^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\omega_0^2\begin{pmatrix} 2 & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \tag{8}$$

So we look for charactistic solution

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \vec{E}e^{-i\omega t} \tag{9}$$

Leading to the matrix equation

$$\begin{pmatrix} 2(-\omega^2 + \omega_0^2) & -r\omega^2 \\ -r\omega^2 & (-r^2\omega^2 + r\omega_0^2) \end{pmatrix}\vec{E} = 0 \tag{10}$$

So the non-trivial solutions are when the determinant to this matrix is zero. Evaluating this determinant leads to

$$2(-\omega^2 + \omega_0)^2(-\omega^2 + \omega_0^2/r) - \omega^4 = 0 \tag{11}$$

So

$$\omega^4 - 2(\omega_0^2 + \omega_0^2/r)\omega^2 + 2\omega_0^2(\omega_0^2/r) = 0 \tag{12}$$

The solutions are

$$\omega^2 = \left(\omega_0^2 + \omega_0^2/r \pm \sqrt{(\omega_0^2 + \omega_0^2/r)^2 - 2\omega_0^2(\omega_0^2/r)}\right) \tag{13}$$

So

$$\omega^2 = \omega_0^2\left(1 + u \pm \sqrt{1 + u^2}\right) \tag{14}$$

where $u = 1/r$

(c) In the limit that $r \to 0$ the center of mass swings comparatively slowly at expected freuency $\omega^2 = \omega_0^2$ due to the torque of the gravitational field. The internal oscillations are faster, and oscillates at frequency $\omega^2 = 2g/\ell_2$. One can simply take the limit that $u \to \infty$ yielding the two frequencies

$$\omega^2 \simeq \begin{cases} \omega_0^2 + \mathcal{O}(\omega_0^2/u) \\ \omega_0^2(2u+1) + \mathcal{O}(\omega_0^2/u) \end{cases} \tag{15}$$

2 A scale transformation

(a) An infinitesimal transformation (canonical or not) is defined by the map

$$\boldsymbol{r} \to \boldsymbol{R} = \boldsymbol{r} + \epsilon\, \delta_\epsilon \boldsymbol{r}\,, \tag{16a}$$

$$\boldsymbol{p} \to \boldsymbol{P} = \boldsymbol{p} + \epsilon\, \delta_\epsilon \boldsymbol{p}\,, \tag{16b}$$

where $\delta_\epsilon \boldsymbol{r}$ and $\delta_\epsilon \boldsymbol{p}$ are functions of $\boldsymbol{r}$ and $\boldsymbol{p}$, and ϵ is an infinitesimal parameter. Show that if transformation leaves the Hamiltonian unchanged, and is canonical with generator $G(\boldsymbol{r},\boldsymbol{p})$, then $G(\boldsymbol{r},\boldsymbol{p})$ is constant in time.

(b) Show that the infinitesimal scale transformation

$$\boldsymbol{r} \to \boldsymbol{R} = (1+\epsilon)\boldsymbol{r}\,, \tag{17a}$$

$$\boldsymbol{p} \to \boldsymbol{P} = \frac{\boldsymbol{p}}{(1+\epsilon)} \simeq (1-\epsilon)\boldsymbol{p}\,, \tag{17b}$$

is an infinitesimal canonical transformation. Determine the generator $G(\boldsymbol{r},\boldsymbol{p})$ of this transformation.

Now consider the motion of a particle of mass m in the potential

$$U(x,y,z) = \frac{\boldsymbol{k}\cdot\boldsymbol{r}}{r^3} = \frac{kz}{(x^2+y^2+z^2)^{3/2}}\,, \tag{18}$$

where $\boldsymbol{r} = (x,y,z)$ is the position vector, and $\boldsymbol{k} = k\,\hat{\mathbf{z}}$ is a constant vector in the z direction[1]

(c) Use the transformation in Eq. (17) (as opposed to direct use of the equations of motion for $\dot{\boldsymbol{r}}$ and $\dot{\boldsymbol{p}}$ in this potential) to determine how the generator $G(\boldsymbol{r},\boldsymbol{p})$ from part (b) depends on time for a trajectory in this potential with energy E.

[1] This is the electrostatic potential energy between a charged particle and pointlike electric dipole at the origin.

(a) In a canonical transformation generated by $G(\boldsymbol{r}, \boldsymbol{p})$ we have

$$\begin{aligned} \delta_\epsilon r^i =& \frac{\partial G}{\partial p_i}\,, && (19) \\ \delta_\epsilon p_i =& -\frac{\partial G}{\partial r^i}\,. && (20) \end{aligned}$$

If the Hamiltonian is unchanged we have

$$H(\boldsymbol{r} + \epsilon\delta_\epsilon \boldsymbol{r}, \boldsymbol{p} + \epsilon \delta_\epsilon \boldsymbol{p})\,. = H(\boldsymbol{r}, \boldsymbol{p}) \tag{21}$$

So exapanding this expression to order epsilon we find

$$\epsilon\left(\frac{\partial H}{\partial r^i}\frac{\partial G}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial G}{\partial r^i}\right) = 0\,. \tag{22}$$

But this is (ϵ times) the Poisson bracket of $\{H, G\}$, and so $\{H, G\} = 0$. Recalling the Hamilton's equation of motion for any observable O

$$\dot{O} = \{O, H\}\,, \tag{23}$$

we have

$$\dot{G} = 0\,. \tag{24}$$

(b) The transformation is clearly canonical: The Poisson brackets are unchaged

$$\{P_i, R^j\} = \{\frac{1}{1+\epsilon}p_i, (1+\epsilon)r^i\} = \{p_i, q^j\} = \delta_i^j \tag{25}$$

So we try $G = \boldsymbol{r} \cdot \boldsymbol{p}$. Then

$$\begin{aligned} r^i \to& r^i + \epsilon\frac{\partial G}{\partial p_i} && (26) \\ p_i \to& p_i - \epsilon\frac{\partial G}{\partial r^i} && (27) \end{aligned}$$

which works

$$\begin{aligned} r^i \to& (1+\epsilon) r^i && (28) \\ p_i \to& (1-\epsilon) p_i && (29) \end{aligned}$$

(c) Under the scale transformation in Eq. (17) we have

$$H \to \frac{1}{(1+\epsilon)^2} H\,. \tag{30}$$

So the change in H under the action of G is

$$\delta H = -2\epsilon H\,. \tag{31}$$

For any observable O the change in O under the action of G is

$$\delta O = \epsilon\{O, G\}\,. \tag{32}$$

Applying this to the Hamiltonian we have

$$\delta H = \epsilon\{H, G\} = -\dot{G}\,, \tag{33}$$

So combining the ingredients we have

$$\dot{G} = 2H\ , \tag{34}$$

and thus

$$G = 2Et + \mathrm{const}\,. \tag{35}$$

3 Torque on a box

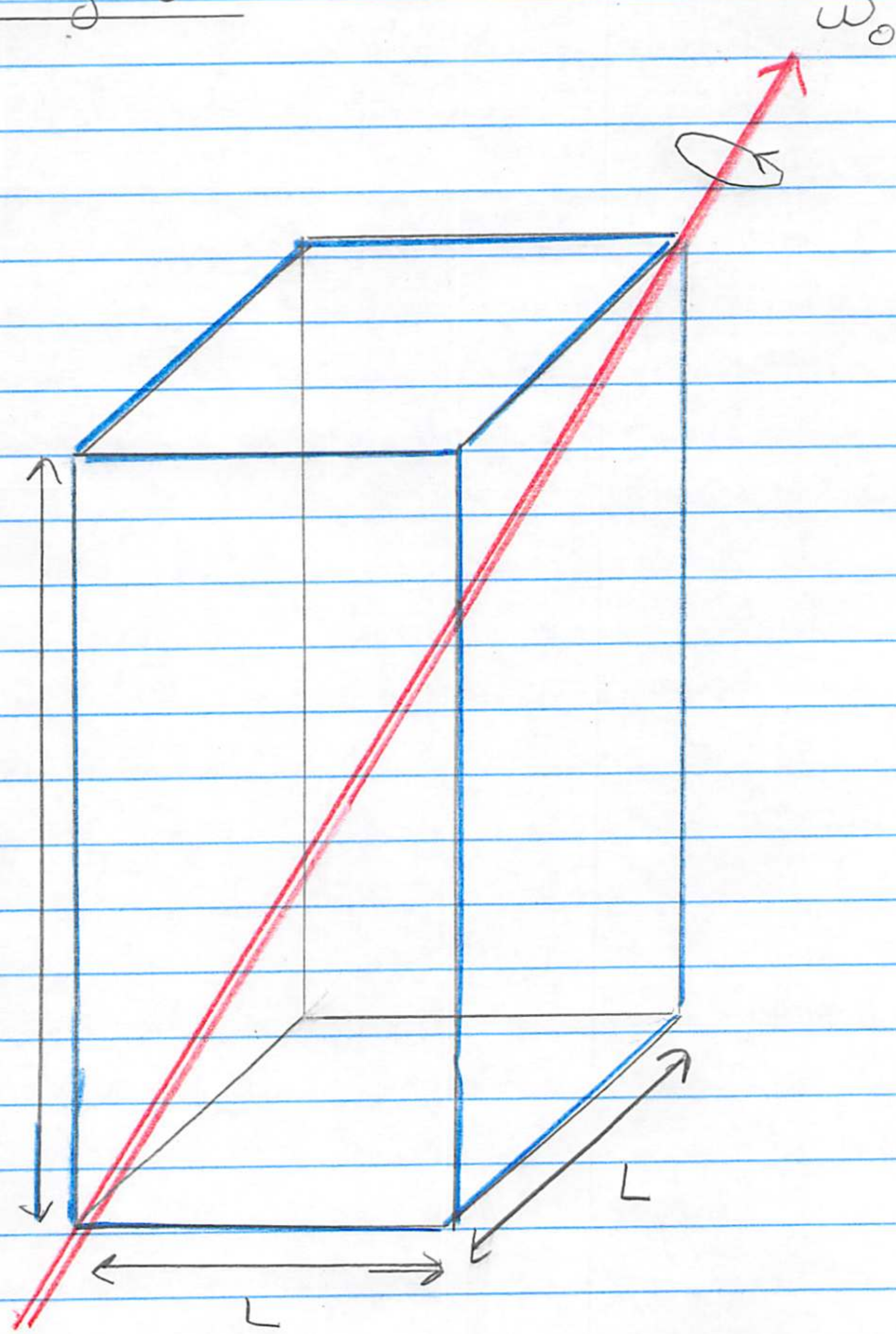
Consider a solid box of mass m and dimension $L, L, 2L$ (see figure).

(a) Compute all components of the moment of inertia tensor around center of mass.

(b) The box is rotated with constant angular frequency ω_0 around its diagonal. At $t = 0$ the box is oriented as shown in the figure. Compute the angular momentum and kinetic energy at this time.

(c) Compute the torque required (both magnitude and direction) to maintain the box's rotational motion at time $t = 0$. Does the magnitude depend on time?

A Rotating Box



Solution

(a) The principal axes are clearly the x, y, z coordinate system

$$I_{xx} = \int dm(y^2 + z^2) \tag{36}$$

$$I_{yy} = \int dm(x^2 + z^2) \tag{37}$$

$$I_{zz} = \int dm(x^2 + y^2) \tag{38}$$

Working through the first example

$$\begin{aligned} I_{xx} =& \frac{m}{2L^3} \times \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \int_{-L}^{L} dz(y^2 + z^2) && (39) \\ =& \frac{m}{2L^3} \left[L \times \left(\frac{2}{3} \frac{L^3}{8} \right) \times 2L + L \times L \times \frac{2}{3} L^3 \right] && (40) \\ =& mL^2 \left[\frac{1}{12} + \frac{1}{3} \right] && (41) \\ =& \frac{5}{12} mL^2 && (42) \end{aligned}$$

The other integrals work out by analogy

$$I_{yy} = I_{xx}\,, \tag{43}$$

while

$$I_{zz} = mL^2 \frac{2}{12}\,. \tag{44}$$

To summarize we have

$$I = \frac{1}{12} mL^2 \begin{pmatrix} 5 & & \\ & 5 & \\ & & 2 \end{pmatrix} \tag{45}$$

(b) Then the angular momentum in the body axes are

$$L_a = I_{ab} \omega_b \tag{46}$$

The angular velocity is $(\omega_1, \omega_2, \omega_3) = \frac{\omega_0}{\sqrt{6}}(1, 1, 2)$. So we find

$$L = \frac{mL^2 \omega_0}{12\sqrt{6}} \begin{pmatrix} 5 & & \\ & 5 & \\ & & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \frac{mL^2 \omega_0}{12\sqrt{6}} \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix} \tag{47}$$

The kinetic energy is

$$T = \frac{1}{2} \vec{\omega} \cdot \vec{L}\,, \tag{48}$$

yielding

$$T = \frac{mL^2 \omega_0^2}{2 \cdot 12 \cdot 6} (5 + 5 + 8)\,. \tag{49}$$

(c) The torque is the change in angular momentum

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L} \tag{50}$$

So, up to normalization, we need to compute the cross product of

$$\boldsymbol{v} = (1,1,2) \times (5,5,4) \propto \vec{\omega} \times \vec{L} \tag{51}$$

Straightforward manipulations shows this is

$$\boldsymbol{v} = (1,1,2) \times [(5,5,8) - (0,0,4)] \,, \tag{52}$$

$$\begin{aligned} \boldsymbol{v} =& (\boldsymbol{e}_x + \boldsymbol{e}_y) \times (-4\boldsymbol{e}_z) \,, & (53)\\ =& 4\boldsymbol{e}_y - 4\boldsymbol{e}_x \,. & (54) \end{aligned}$$

Inserting the normalization factors we find

$$\begin{aligned} \vec{\tau} =& \frac{mL^2\omega_0}{12\sqrt{6}} \frac{\omega_0}{\sqrt{6}} (4\boldsymbol{e}_y - 4\boldsymbol{e}_x) & (55)\\ =& \frac{mL^2\omega_0^2}{18} (\boldsymbol{e}_y - \boldsymbol{e}_x) \,. & (56) \end{aligned}$$

The magnitude is indepdent of time

$$\tau = \frac{\sqrt{2}}{18} mL^2\omega_0^2 \,. \tag{57}$$

4 A coupled chain of pendulums

Consider a chain of coupled pendulums in the earth's gravitational field. The pendulums are separated by a distance a, and have rods of length ℓ (see figure). The masses at the ends of the pendulums have mass m and are connected by springs of spring constant κ, which are unstretched when the system is at rest. All rods and springs may be considered massless.

(a) Write down the Lagrangian of the system for small angular oscillations

(b) Determine the oscillation frequency $\omega(k)$ for eigenmodes of wavenumber k. Determine the group velocity for $ka \ll 1$, and sketch the result versus k.

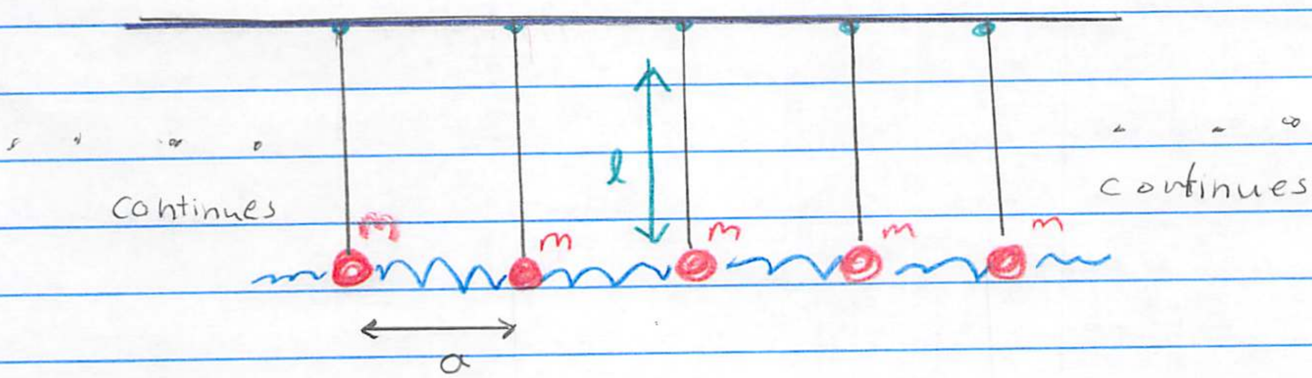
(c) Consider the continuum action

$$S[q(t,x)] = \int dtdx\, \frac{1}{2}\mu(\partial_t q(t,x))^2 - \frac{1}{2}Y(\partial_x q(t,x))^2 - \frac{1}{2}\gamma^2 q^2(t,x)\,. \tag{58}$$

where μ, Y, and γ are constants. Determine the equations of motion.

(d) Find the dispersion curve $\omega(k)$ for the plane wave solutions $Ae^{ikx-i\omega(k)t}$ to the continuum equations of part (c). What should the continuum parameters μ, Y, and γ be to reproduce the discrete results of part (b) at small k.

Coupled Pendulums



(a) The equilibrium position of the j-th oscillator is $(x_j, y_j) = (ja, 0)$. The angles all fluctuating by small amounts. The change in postitions

$$\delta x_j = \ell\,\theta_j \tag{59}$$

$$\delta y_j = \frac{1}{2}\ell\,\theta_j^2 \tag{60}$$

The Lagrangian is

$$L = \sum_j \frac{1}{2}m\left(\frac{\delta x_j}{dt}\right)^2 - mg\delta y_j - \frac{1}{2}\kappa(\delta x_j - \delta x_{j-1})^2 \tag{61}$$

This expands to

$$L = \sum_j \frac{1}{2}m\ell^2\dot\theta_j^2 - \frac{1}{2}mg\ell\theta_j^2 - \frac{1}{2}\kappa\ell^2(\theta_j - \theta_{j-1})^2 \tag{62}$$

Writing out the equation of motion we find

$$m\ell^2\ddot\theta_j = -mg\ell\theta_j - \kappa\ell^2(\theta_j - \theta_{j-1}) + \kappa\ell^2(\theta_{j+1} - \theta_j) \tag{63}$$

Dividing by $m\ell^2$ we find

$$\ddot\theta_j = -\Omega^2\theta_j + \omega_0^2\left(\theta_{j+1} - 2\theta_j + \theta_j\right) \tag{64}$$

where $\Omega^2 = g/\ell$ and $\omega_0^2 = \kappa/m$.

(b) Now we substitute $\theta_j = Ae^{ikx_j - i\omega t}$ into Eq. (64) . Note that

$$\theta_{j+1} = Ae^{ik(x_j+a)-i\omega t} = e^{ika}\,Ae^{i(kx_j - i\omega t)}\,. \tag{65}$$

Thus minor manipulations lead to

$$-\omega^2 = -\Omega^2 + \omega_0^2(e^{ika} - 2 + e^{-ika})\,. \tag{66}$$

And so, using $4\sin^2(ka/2) = 2 - 2\cos(ka)$, we find that

$$\omega^2 = \Omega^2 + 4\omega_0^2\sin^2(ka/2)\,. \tag{67}$$

For small k we find

$$\omega(k) = \pm\sqrt{\Omega^2 + v_0^2k^2}\,, \tag{68}$$

where $v_0 \equiv \omega_0 a$. The group velocity is

$$\frac{d\omega}{dk} = \pm\frac{v_0^2k}{\sqrt{\Omega^2 + v_0^2k^2}} \tag{69}$$

This is the dispersion curve of massive relativistic particle of mass m and momentum p if one identifies $\Omega = (mc^2)$, $v_0 = c$, and $p = k$.

(c) From the Euler Lagrange equations

$$- \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)} \right) + \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{70}$$

We find

$$\partial_t(\mu \partial_t q) - \partial_x(Y \partial_x q) + \gamma^2 q = 0\,. \tag{71}$$

(d) Substiting the ansatz $Ae^{ikx-i\omega t}$ we find

$$- \mu\omega^2 + Yk^2 + \gamma^2 q^2 = 0 \tag{72}$$

and the dispersion curve is

$$\omega = \pm\sqrt{\gamma^2 + \frac{Y}{\mu}k^2} \tag{73}$$

So we want to take

$$\frac{Y}{\mu} \Rightarrow v_0^2 = \frac{\kappa a^2}{m}\,, \tag{74}$$

$$\gamma^2 \Rightarrow \Omega^2 = \frac{g}{\ell}\,, \tag{75}$$

in order that the dispersion curves match. Finally one would (if needed) set

$$\mu \Rightarrow \frac{m\ell^2}{a}\,, \tag{76}$$

so that

$$\int dx \mu \, (\partial_t q)^2 \simeq \sum_j m\ell^2 \, (\partial_t \theta_j)^2\,. \tag{77}$$