Physics 501: Classical Mechanics

Final Exam

Stony Brook University

Fall 2020

General Instructions:

You may use one page (front and back) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Problem 1. A symmetric Lagrangian

Consider the Lagrangian

$$L = \frac{m(t)}{2} \frac{\dot{x}^2 + \dot{y}^2 + 2\omega(x\dot{y} - y\dot{x})}{x^2 + y^2} \tag{1}$$

with $m > 0$

(a) Determine the Hamiltonian $H(x, p_x, y, p_y)$ of the system.

(b) Find the infinitesimal transformations generated by $G_1 = \boldsymbol{r} \cdot \boldsymbol{p}$ and $G_2 = (xp_y - yp_x)$ and describe the transformations physically.

(c) Show that G_1 and G_2 are constant in time using the results of (b).

Solution:

(a) Ok we start

$$H = p_x\dot{x} + p_y\dot{y} - L\,. \tag{2}$$

$$p_x = \frac{m(\dot{x} - \omega y)}{x^2 + y^2}\,, \tag{3}$$

$$p_y = \frac{m(\dot{y} + \omega x)}{x^2 + y^2}\,. \tag{4}$$

So we determine

$$\dot{x} = \frac{(x^2 + y^2)p_x - \omega y}{m}\,, \tag{5}$$

$$\dot{y} = \frac{(x^2 + y^2)p_y + \omega x}{m}\,. \tag{6}$$

Substituting these expressions in we find

$$H = \frac{(x^2 + y^2)}{2m}\left[(p_x - \frac{m\omega y}{x^2 + y^2})^2 + (p_y + \frac{m\omega x}{x^2 + y^2})^2\right]\,. \tag{7}$$

Expanding it out with $\rho = \sqrt{x^2 + y^2}$ we have

$$H = \frac{\rho^2}{2m}\left[p_x^2 + p_y^2 + m\omega\frac{1}{\rho^2} - \frac{m\omega}{\rho^2}\,(p_x y - p_y x)\right]\,, \tag{8}$$

which is also OK.

(b) The transformations are from G_1 gives

$$\boldsymbol{r} \to (1 + \lambda)\boldsymbol{x}\,, \tag{9}$$

$$\boldsymbol{p} \to (1 - \lambda)\boldsymbol{p}\,. \tag{10}$$

More generally (see homework) this is an approximation for λ small to

$$\boldsymbol{r} \to a\boldsymbol{r}, \tag{11}$$

$$\boldsymbol{p} \to \frac{\boldsymbol{p}}{a}, \tag{12}$$

with $a = 1 + \lambda$. This is the generator of conformal transformations.

From $G_2 = xp_y - yp_x$ we have

$$x \to x - \lambda y, \tag{13}$$

$$y \to y + \lambda x, \tag{14}$$

$$p_x \to p_x - \lambda p_y, \tag{15}$$

$$p_y \to p_y + \lambda p_x. \tag{16}$$

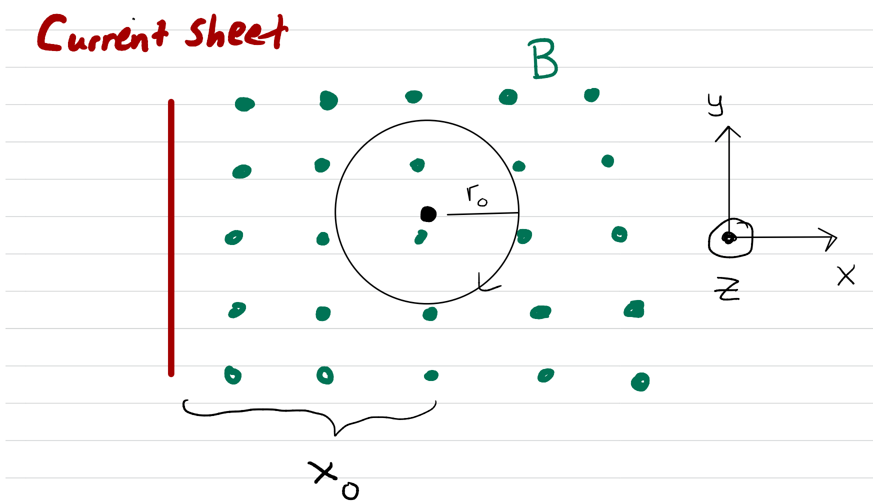
This is the generator of rotations

(c) First lets discuss rotations. Under rotations ρ^2 and $p_x^2 + p_y^2$ are clearly invariant. The quantity $L_z = xp_y - yp_z$ is the angular momentum around in the z direction (the only direction), and this is also clearly invariant under rotations around z. Thus the full Hamiltonian is invariant under the transformation generated by G_2 and thus G_2 is constant in time.

Then lets discuss the recalling. Under the rescaling $\rho \to a\rho$, $\boldsymbol{r} \to a\boldsymbol{r}$, and $\boldsymbol{p} \to \boldsymbol{p}/a$. From this rescaling we see that the Hamiltonian is actually invariant under this transformation, and thus $G_1 = \boldsymbol{r} \cdot \boldsymbol{p}$ is also constant in time.

Problem 2. A slowly changing magnetic field

Consider the circular orbits in the xy plane with $x > 0$ of a particle mass m and charge q in a constant and uniform magnetic field B in the z direction. (This magnetic field could be created by a sheet of current in the yz plane at $x = 0$ as shown below.)



(a) Use the Hamiltonian formulation to determine the radius and angular frequency of the circular orbits. Relate the center of the circular orbit to the canonical momenta of the problem. Use the gauge

$$\boldsymbol{A} = B(0, x, 0)\,.$$

It is useful to define the cyclotron frequency[1], $\omega_B = qB/mc$.

Now imagine that starting at $t = 0$ the strength of the magnetic field is slowly increased from its initial value of $B_0 \equiv B(0)$.

(b) If the original orbit has radius r_0 and is centered at $\boldsymbol{x}_0 = (x_0, 0, 0)$ with $x_0 > 0$, determine how the radius and the center of the circular orbits change as $B(t)$ is slowly increased. Describe your results qualitatively by drawing a sketch.

[1] We have given cyclotron frequency in Gaussian units. In SI units $\omega_B = qB/m$.

Solution:

(a) The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{q}{c}\left(A_x\dot{x} + A_y\dot{y}\right), \tag{17}$$

Or

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + m\omega_B(t)x\dot{y}\,. \tag{18}$$

We construct the Hamiltonian which matches onto the general form discussed in class:

$$H = \frac{1}{2m}\left(p_x^2 + (p_y - m\omega_B x)^2\right)\,. \tag{19}$$

The equation of motion for p_y is cyclic in character

$$p_y = \text{const}, \tag{20}$$

Thus the effective Hamiltonian for the motion in x is then

$$H_{\text{eff}} = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_B^2\left(x - \frac{p_y}{m\omega_B}\right)^2, \tag{21}$$

which is a shifted harmonic oscillator in x. The harmonic motion is around

$$\frac{p_y}{m\omega_B}\,. \tag{22}$$

which determines the center of the circle.

The radius of the circle is determined by the energy of the $1D$ problem We have that the turning points of the x motion for the equivalent $1d$ problem determines the radius

$$\epsilon = \frac{1}{2}m\omega_B^2 r_0^2\,, \tag{23}$$

or

$$r_0 = \sqrt{\frac{2\epsilon}{m\omega_B^2}}\,. \tag{24}$$

(b) Then using the theory of adiabatic invariants we have

$$I = \oint p_x dx\,, \tag{25}$$

is constant. We have from lecture that the adiabatic invariant for the SHO is

$$I = \frac{\epsilon}{\omega_B}\,. \tag{26}$$

So we can express the radius and center as

$$r(t) = \sqrt{\frac{2I}{m\omega_B}}\,, \qquad x_0 = \frac{p_y}{m\omega_B}\,, \tag{27}$$

Since I and p_y are adiabatically constant and constant respectively we find:

$$r(t) = r(0)\sqrt{\frac{B(0)}{B(t)}}\,, \qquad x_0(t) = x_0(0)\frac{B(0)}{B(t)}\,. \tag{28}$$

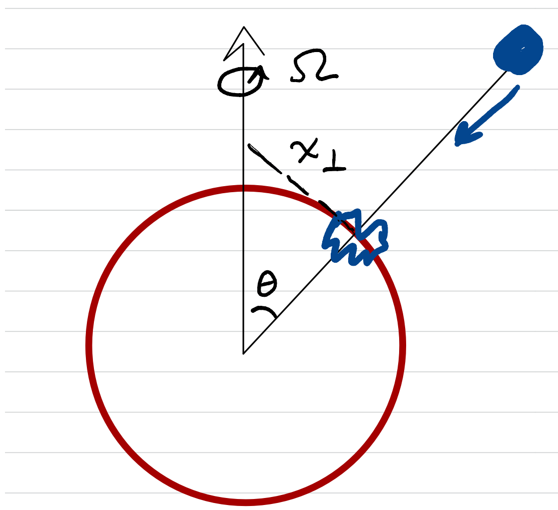
Discussion: Interpreting the result, the radius shrinks keeping the flux fixed:

$$\pi r(t)^2 B(t) = \text{const}\,, \tag{29}$$

and the circular orbits move closer to $x = 0$.

Problem 3. Impact on a distant planet

Consider a distant planet as sphere of mass m and radius r which is initially rotating with angular velocity Ω. A meteor also of mass m strikes the planet head on at angle θ relative to the rotation axis, and becomes stuck to the surface of the planet:



(a) Determine the moment of inertia tensor around the new center of mass of the sphere+meteor after the impact. Take the z axis as the line connecting the center of the sphere and the meteor, and the x-axis parallel to $x_\perp$ line as shown in the figure.

You should find the form

$$I_{ab} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_x & 0 \\ 0 & 0 & I_z \end{pmatrix}, \tag{30}$$

which can be used in part (b) below.

(b) Determine the post-impact components of the angular velocity in the body frame as a function of time. Give a physical interpretation of the your time dependent angular velocity by drawing a careful sketch to explain your result.

Solution:

(a) The new center of mass is shifted to halfway between the center of the sphere and the meteor. We need to find the moment of inertia of the sphere around this point, and add to it the moment of inertia of the meteor around this point.

We use the parallel axis theorem first to find the moment of inertia of the sphere at the halfway point. The shift is $d_a = (0,0,r/2)$ and in the z direction

$$
\begin{aligned}
I_{\text{sphere}}^{\text{newcm}} =& I_{\text{sphere}}^{\text{oldcm}} + m\left(d^2\delta_{ab} - d_a d_b\right)\,, && (31)\\
=& \frac{2}{5}mr^2 \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \frac{mr^2}{4}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\,, && (32)\\
=& \frac{mr^2}{20}\begin{pmatrix}13&0&0\\0&13&0\\0&0&8\end{pmatrix}\,. && (33)
\end{aligned}
$$

We should add to this the moment of inertia from the meteor this is just a point like object of mass m and position $r_a = (0,0,r/2)$ which yields

$$
\begin{aligned}
I_{\text{meteor}} =& m\left(r^2\delta_{ab} - r_a r_b\right) && (34)\\
=& \frac{mr^2}{4}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\,. && (35)
\end{aligned}
$$

So adding together these two contributions we have finally

$$
I = \frac{mr^2}{20}\begin{pmatrix}18&0&0\\0&18&0\\0&0&8\end{pmatrix}\,. \tag{36}
$$

(b) To find the the subsequent motion we use the Euler equations for torque free motion.

$$
\left(\frac{d\boldsymbol{L}}{dt}\right)_r + \boldsymbol{\omega}\times\boldsymbol{L} = 0\,. \tag{37}
$$

The angular momentum before and after impact is the same, which fixes the initial conditions for this differential equation.

More explicitly the Euler equations are

$$
\begin{aligned}
I_1\dot{\omega}_1 + \omega_2 L_3 - \omega_3 L_2 =& 0\,, && (38)\\
I_2\dot{\omega}_2 + \omega_3 L_1 - \omega_1 L_3 =& 0\,, && (39)\\
I_3\dot{\omega}_3 + \omega_1 L_2 - \omega_2 L_1 =& 0\,. && (40)
\end{aligned}
$$

In the current context these are

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_1) = 0\,, \tag{41}$$
$$I_2\dot{\omega}_2 + \omega_2\omega_3(I_1 - I_3) = 0\,. \tag{42}$$
$$I_3\dot{\omega}_3 = 0\,. \tag{43}$$

These imply that ω_3 is constant and that ω_1 and ω_2 satisfy

$$\dot{\omega}_1 = -\frac{\Delta I}{I_3}\omega_3\omega_2\,, \tag{44}$$
$$\dot{\omega}_2 = \frac{\Delta I}{I_3}\omega_3\omega_1\,. \tag{45}$$

So we find, defining $z = \omega_1 + i\omega_2$, that

$$\dot{z} = i\gamma z\,, \qquad \gamma \equiv \frac{\Delta I}{I_3}\omega_3\,, \tag{46}$$

The solution to this equation is

$$z = \omega_{10}e^{i\gamma t}, \tag{47}$$

where we have anticipated that at time $t = 0$ there is no angular velocity in the y direction.

The constants of motion, ω_3 and ω_{10}, are given by the conservation of angular momentum. The angular momentum of the initial rotating sphere projected onto the new x, y, z axes is

$$\boldsymbol{L} = I_0\Omega\cos\theta\,\hat{\boldsymbol{z}} + I_0\Omega\sin\theta\,\hat{\boldsymbol{x}}\,, \tag{48}$$

where $I_0 = 2mr^2/5$. At $t = 0$ or solution gives

$$\boldsymbol{L} = I_3\omega_3\,\hat{\boldsymbol{z}} + I_1\omega_{10}\hat{\boldsymbol{x}}\,. \tag{49}$$

Comparison of these two expressions shows that

$$\omega_3 = \frac{I_0}{I_3}\Omega\cos\theta, \tag{50}$$

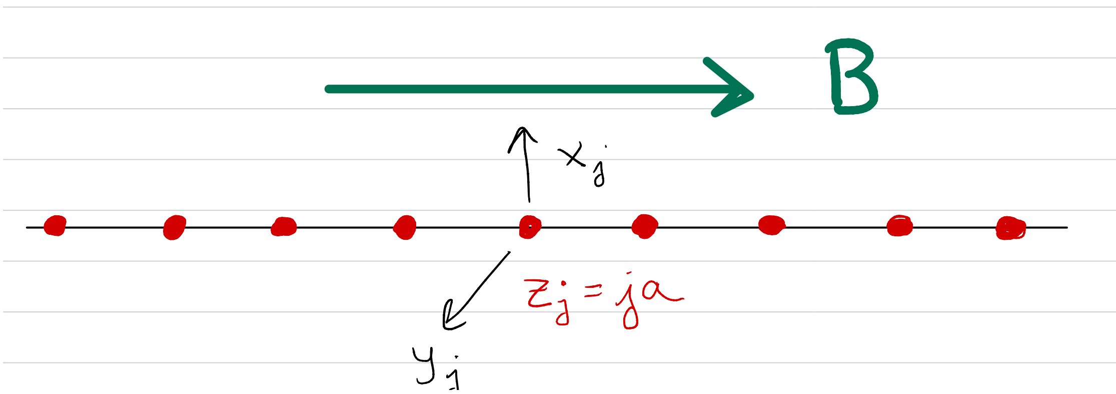
while

$$\omega_{10} = \frac{I_0}{I_1}\Omega\sin\theta. \tag{51}$$

The motion is a kind of wobbling like motion, similar to the rotating plate example discussed in class.

Problem 4. A chain of oscillators in a magnetic field

Consider an infinite linear chain of oscillators at lattice sites $z_j = ja$ consisting of particles of mass m and charge e. The particles are free to move only in the transverse x, y directions, and the displacements at lattice site j are denoted $\boldsymbol{r}_j = (x_j, y_j)$. The interaction between the masses is nearest neighbor, and takes the form $U = \frac{1}{2}\gamma|\boldsymbol{r}_j - \boldsymbol{r}_{j-1}|^2$. Neglect the Coulomb interaction between the particles. The particles sit in a constant magnetic field directed in the z direction, $\boldsymbol{B} = B\hat{\mathbf{z}}$.



(a) Write down the Lagrangian of the system and compute the equations of motion.
Hint: Take the gauge

$$\boldsymbol{A} = \frac{B}{2}(-y, x, 0), \tag{52}$$

It is also helpful to use the variables $\omega_0^2 = \gamma/m$ and the cyclotron frequency[2], $\omega_B = qB/mc$.

(b) Determine the normal modes of oscillations and the associated eigenfrequencies. It is helpful to consider $x \pm iy$.

(c) Sketch the dispersion curve $\omega(k)$ for each independent mode for $ka \ll 1$. Interpret your results in the limit $k \to 0$.

(d) Consider the complex field $\psi(t,z) \equiv q_1(t,z) + iq_2(t,z)$ where q_1 and q_2 are real fields. The action for ψ is

$$S[q_1, q_2] = \int dtdz\, \frac{1}{2}\partial_t\psi\partial_t\psi^* - \frac{1}{2}C_1\partial_z\psi\partial_z\psi^* + \frac{1}{2}C_2\left(\psi^* i\partial_t\psi - \psi i\partial_t\psi^*\right), \tag{53}$$

where C_1, C_2 are real constants.

(i) Determine the equations of motion for ψ by varying the action.
Hint: You can determine the equations of motion ψ from the equation of motion of q_1 and q_2.

(ii) How should the constants C_1 and C_2 be chosen if this action is to reproduce the dispersion curves of part (c).

[2]We have given cyclotron frequency in Gaussian units. In SI units $\omega_B = qB/m$.

Solution:

(a) (a) The Lagrangian is

$$L = \sum_j \frac{1}{2} m \left(\dot{x}_j^2 + \dot{y}_j^2 \right) - \frac{1}{2}\gamma (x_j - x_{j-1})^2 - \frac{1}{2}\gamma (y_j - y_{j-1})^2 + \frac{q}{c} \left(A_x \dot{x} + A_y \dot{y} \right), \tag{54}$$

$$= \sum_j \frac{1}{2} m \left(\dot{x}_j^2 + \dot{y}_j^2 \right) - \frac{1}{2}\gamma (x_j - x_{j-1})^2 - \frac{1}{2}\gamma (y_j - y_{j-1})^2 + \frac{q}{2c} \left(x_j \dot{y}_j - y_j \dot{x}_j \right). \tag{55}$$

Then the equation of motion is

$$\ddot{x}_j = -\omega_0^2 (x_j - x_{j-1}) + \omega_0^2 (x_{j+1} - x_j) + \omega_B \dot{y}_j, \tag{56}$$

$$\ddot{y}_j = -\omega_0^2 (y_j - y_{j-1}) + \omega_0^2 (y_{j+1} - y_j) - \omega_B \dot{x}_j, \tag{57}$$

where $\omega_B = qB/mc$.

(b) Then substituting

$$x_j = X e^{ikz_j - i\omega t}, \tag{58}$$

$$y_j = Y e^{ikz_j - i\omega t}, \tag{59}$$

we find

$$-\omega^2 X = -\omega_0^2 X (2 - 2\cos(ka)) - i\omega_B \omega Y, \tag{60}$$

$$-\omega^2 Y = -\omega_0^2 Y (2 - 2\cos(ka)) + i\omega_B \omega X. \tag{61}$$

The eigen-system is easily diagonalized by switching to $q = X + iY$ and bar $\bar{q} = X - iY$ we find

$$-\omega^2 q = -\omega_0^2 (2 - 2\cos(ka)) q - \omega_B \omega q, \tag{62}$$

$$-\omega^2 \bar{q} = -\omega_0^2 (2 - 2\cos(ka)) \bar{q} + \omega_B \omega \bar{q}. \tag{63}$$

Thus $q_s = X + siY$ with $s = \pm$ are the eigen modes. The associated eigen frequencies are the roots of

$$\omega^2 - s\omega\omega_B - \omega_0^2 (2 - 2\cos(ka)) = 0, \tag{64}$$

or

$$\omega_s = \tfrac{1}{2} s\omega_B \pm \sqrt{(s\tfrac{1}{2}\omega_B)^2 + \omega_0^2 (2 - 2\cos(ka))}. \tag{65}$$

(c) At small k where $v_0 = \omega_0 a$

$$\omega = -\tfrac{1}{2} s\omega_B \pm \sqrt{(\tfrac{1}{2}\omega_B)^2 + (v_0 k)^2}, \tag{66}$$

which is sketched in the figure. At $k = 0$ we have $\omega = 0$ which corresponds to a static string, and $\omega = \omega_B$ which corresponds to uniform circular motion of the string.

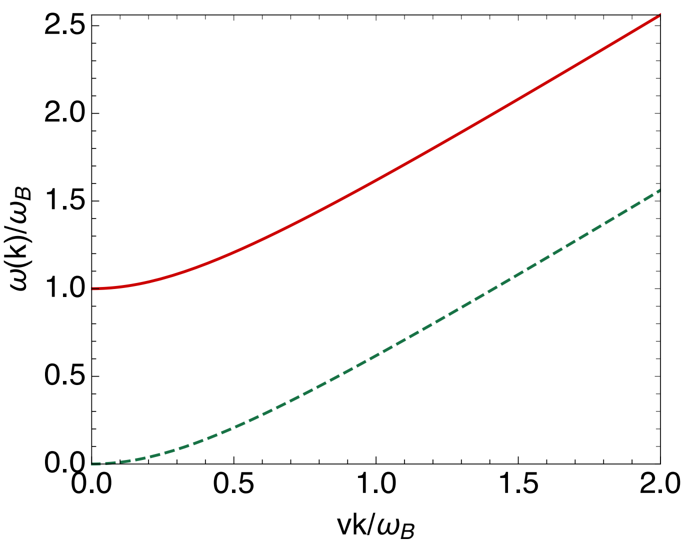


Figure 1: The dispersion curve

(d) (i) Writing this out using

$$\partial\psi\partial\psi^* = \partial q_1\partial q_2 + \partial q_2\partial q_2\,, \tag{67}$$

$$\frac{1}{2}\left(\psi^* i\partial_t\psi - \psi i\partial_t\psi^*\right) = -\left(q_1\dot{q}_2 - q_2\dot{q}_1\right)\,, \tag{68}$$

gives the action

$$S = \int dtdz\left[\tfrac{1}{2}(\partial_t q_1)^2 - \tfrac{1}{2}C_1(\partial_z q_1)^2 + \tfrac{1}{2}(\partial_t q_2)^2 - \tfrac{1}{2}C_1(\partial_z q_2)^2 - C_2(q_1\partial_t q_2 - q_2\partial_t q_1)\right]\,. \tag{69}$$

So varying the action gives the equation o f motion

$$\frac{\delta S}{\delta q_1} = -\partial_t^2 q_1 + C_1\partial_z^2 q_1 - C_2\partial_t q_2 = 0\,, \tag{70}$$

$$\frac{\delta S}{\delta q_2} = -\partial_t^2 q_2 + C_1\partial_z^2 q_2 + C_2\partial_t q_1 = 0\,. \tag{71}$$

The equation of motion for ψ is

$$\frac{\delta S}{\delta q_1} + i\frac{\delta S}{\delta q_2} = -\partial_t^2\psi + C_1\partial_z^2\psi + C_2 i\partial_t\psi = 0\,. \tag{72}$$

(ii) Substituting

$$\psi = Ae^{ikz-i\omega t}\,, \tag{73}$$

The dispersion curve is

$$\omega^2 - C_2\omega - C_1k^2 = 0\,. \tag{74}$$

Comparison with the low-k limit of Eq. (65)

$$\omega^2 + s\omega_B\omega - v_0^2k^2 = 0\,, \tag{75}$$

shows that we should take

$$C_2 = -\, s\omega_B\,, \tag{76}$$

$$C_1 = v_0^2\,. \tag{77}$$