1.3 The Hamiltonian Formalism, the Routhian, and the Legendre Transform

The Hamiltonian formalism: basic version

- Let the Lagrangian be a convex function of the velocity $v_q \equiv \dot{q}$. In one dimension this means that the momentum $p = \partial L/\partial v_q$ is an increasing function of the velocity $v_q \equiv \dot{q}$, i.e $\partial^2 L/\partial \dot{q}^2 > 0$. This means there is one value of the velocity for given momentum p, $\dot{q}(p)$. Clearly $L \propto v^2$ is convex.

  In higher dimensions we require that $\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j$ is a positive definite matrix. This means that for a given value of p_i there is a unique value of the velocity vector $v_q^i \equiv \dot{q}^i(p)$ at fixed q.

- With convex function $L(\dot{q})$ a Legendre transform useful, and trades the velocity dependence of the Lagrangian dependence for the momentum dependence p of the Hamiltonian

  First note

$$dL = p\, d\dot{q} + \underbrace{\frac{\partial L}{\partial q} + \frac{\partial L}{\partial t} dt}_{\text{“spectators”}} \tag{1.51}$$

  We can trade the $d\dot{q}$ for dp by looking at $L - p\dot{q}$, or, as is conventional, minus this quantity. Thus we define

$$H(p, q, t) = p\, \dot{q}(p) - L(\dot{q}(p), q, t) \tag{1.52}$$

  where $\dot{q}(p)$ is determined from p at fixed q and t, i.e. we must invert the relation

$$p = \frac{\partial L(\dot{q}, q, t)}{\partial \dot{q}} \;\Rightarrow\; \text{determines } \dot{q}(p) \tag{1.53}$$

  We have (do it yourself!)

$$dH(p, q, t) = \dot{q}\, dp - \underbrace{\left(\frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial t} dt\right)}_{\text{“spectators”}} . \tag{1.54}$$

  Thus we have

$$\frac{\partial H}{\partial p} = \dot{q} \qquad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{1.55}$$

  were L is a function $\dot{q}$ and H is a function of the corresponding p. You should be able to show that these results (together with the Euler-Lagrange equations) yield Hamilton's equations of motion:

$$\frac{dq}{dt} = \frac{\partial H(q, p, t)}{\partial p} \tag{1.56}$$

$$\frac{dp}{dt} = -\frac{\partial H(q, p, t)}{\partial q} \tag{1.57}$$

- When more variables are around then we simply sum over the $p_i \dot{q}^i$ term

$$H(p, q, t) = \sum_i p_i \dot{q}^i(p) - L(\dot{q}(p), q, t) \tag{1.58}$$

  and the equation of motion are

$$\frac{dq^i}{dt} = \frac{\partial H(q, p, t)}{\partial p_i} \tag{1.59}$$

$$\frac{dp_i}{dt} = -\frac{\partial H(q, p, t)}{\partial q^i} \tag{1.60}$$

- The total derivative of the Hamiltonian satisfies

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{1.61}$$

so that if H is not an explicitly function of time then it is constant.

- For a (rather general) Lagrangian of the form

$$L = \frac{1}{2} a_{ij}(q)\dot{q}^i\dot{q}^j + b_i(q)\dot{q}^i - U(q)\,, \tag{1.62}$$

the momenta and velocities are related via

$$p_i = a_{ij}\dot{q}^j + b_i\,, \qquad \dot{q}^i = (a^{-1})^{ij}(p_j - b_j)\,. \tag{1.63}$$

The Hamiltonian is

$$H(p, q, t) = \frac{1}{2}(a^{-1})^{ij}(p_i - b_i)(p_j - b_j) + U(q)\,. \tag{1.64}$$

This should be compared to the hamiltonian function in (1.46). The Hamiltonian is a function of the b_i, while the hamiltonian function is not. The Hamiltonian and hamiltonian function return the same value at corresponding points where $\dot{q} = \dot{q}(p)$, but have different functional forms.

The action principle

- The Hamiltonian can be used in the action principle to determine the equation of motion. The action takes a path in p, q space $(p_i(t), q^i(t))$ and returns a number

$$S[p(t), q(t), t] = \int \mathrm{d}t\, \left(p_i\, \dot{q}^i - H(p, q, t)\right) \tag{1.65}$$

We note $p_i\dot{q}^i - H = L$ at corresponding points. Varying the action with $p_i(t)$ and $q^i(t)$ separately (keeping the ends fixed) gives the Hamiltonian equation of motion. By doing this variation you should be able to show that

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}\,, \tag{1.66}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}\,. \tag{1.67}$$

The Routhian

- It is often convenient to Legendre transform with respect to some of the coordinates. (This is usually convenient for the cyclic coordinates).

Suppose we have two coordinates x and y, with Lagrangian $L(\dot{x}, x, \dot{y}, y)$. If we Legendre transform with respect to $\dot{x}$ (replacing it with p_x), but leave $\dot{y}$ alone:

$$R(p_x, x, \dot{y}, y) \equiv p_x\, \dot{x}(p_x) - L(\dot{x}(p_x), x, \dot{y}, y)\,, \tag{1.68}$$

then R (known as the Routhian) acts like a Hamiltonian for (p_x, x), but a Lagrangian[2] for $(\dot{y}, y)$. You should be able to show that

$$\frac{dx}{dt} = \frac{\partial R}{\partial p_x} \tag{1.69}$$

$$\frac{dp_x}{dt} = -\frac{\partial R}{\partial x} \tag{1.70}$$

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{y}}\right) = \frac{\partial R}{\partial y} \tag{1.71}$$

Here, since the variables in R are $p_x, x, \dot{y}$ and y, the partial derivative, $\partial R/\partial y$, means, $(\partial R/\partial y)_{p_x}$. In the Lagrangian setup $L(\dot{x}, x, \dot{y}, y)$, with variables $\dot{x}, x, \dot{y}$ and y, one would have $(\partial L/\partial y)_{\dot{x}}$.

[2] Technically it is actually $-R$ that is Lagrangian for $\dot{y}, y$, due to the fact we are subtracting L when making the Legendre transform in Eq. (1.68). Of course you could have done the following $R = L - p_x\dot{x}$, and then it would be a Lagrangian for y, but $-R$ would be the Hamiltonian for x.

The Legendre Transform as extremization in the presence of an external bias (force)

- Consider the convex function $U(x)$. Its derivative is[3]

$$dU = f_0(x)\, dx \tag{1.72}$$

  Then we define[4]

$$\hat{U}(x, f) = f x - U(x)\,. \tag{1.73}$$

  Then the Legendre transform is the extremum (maximum or minimum) of $\hat{U}(x, f)$ for fixed f, i.e.

$$V(f) = \text{extrm}_x\ (f\, x - U(x))\ . \tag{1.74}$$

  This means that we are to change x until we reach the value $x(f)$ where $\hat{U}$ is a maximum or minimum. The value of $\hat{U}$ at this point is $V(f)$. By differentiation, the extremal point is when $f = dU/dx = f_0(x)$, which must be inverted to determine $x(f)$. Then $V(f) = fx(f) - U(x(f))$.

- We have

$$dU = f(x)\, dx \qquad \text{and} \quad dV = x(f)\, df \tag{1.75}$$

  and a relation between the second derivatives

$$\frac{d^2U}{dx^2}\frac{d^2V}{df^2} = 1 \tag{1.76}$$

- Then inverse Legendre transform returns the back the potential

$$U(x) = \text{extrm}_f\ (f\, x - V(f)) \tag{1.77}$$

  which you should prove for yourself.

- For more degrees of freedom, take $U(x_1, x_2)$ for example, the procedure works similarly. We define

$$V(f_1, f_2) = \text{extrm}_{x_1,x_2}\left(f_1 x^1 + f_2 x^2 - U(x)\right) \tag{1.78}$$

  Then

$$dU = f_1\, dx^1 + f_2\, dx^2 \qquad \text{and} dV = x^1\, df_1 + x^2\, df_2 \tag{1.79}$$

  Note that the matrices of second derivatives

$$U_{ij} \equiv \frac{\partial^2 U}{\partial x^i \partial x^j} \qquad V^{ij} \equiv \frac{\partial^2 V}{\partial f_i \partial f_j} \tag{1.80}$$

  are inverses of each

$$V^{i\ell} U_{\ell j} = \delta^i_j \tag{1.81}$$

[3] Think of $U(x)$ as the spring like potential that a particle feels. Then $f_0(x)$ is the external force that must be *applied* to the system so that the particle is in equilibrium at position x. The "internal" force that the potential gives is $f_{\text{internal}}(x) = -dU/dx$. This internal force must be counterbalanced by the applied force $f_0(x) = -f_{\text{internal}}(x)$.

[4] Referring to to the previous footnote $\hat{U}(x, f)$ is minus the potential in the presence of an applied external force f. In thermodynamics we would define the Legendre transform with $\hat{U} = U - fx$, but the overall sign leads only to minor differences. We follow the mechanics convention, $H = pv_q - L$, with regard to sign.