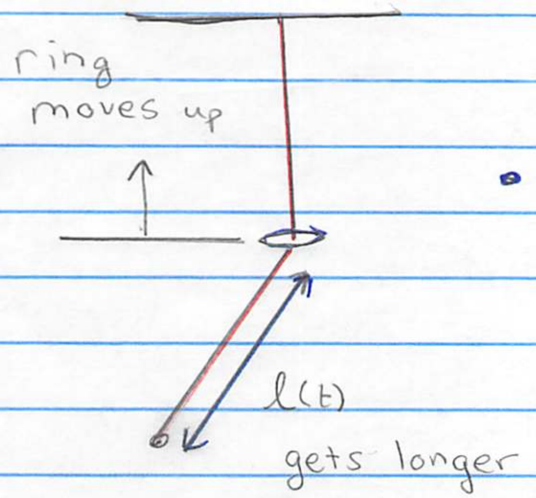
Adiabatic Invariance

- Example



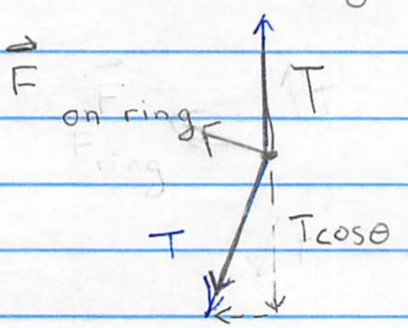
- A string pendulum is passed through a loop. The loop is slowly raised

- The string of the pendulum does work on the ring. E changes in time, but the resonance frequency $\omega = \sqrt{\frac{g}{\ell}}$ also changes, leaving:

$$\frac{E}{\omega} \propto E\sqrt{\ell} \simeq \text{constant}$$

Simple arguement:

There is an equal-and-opposite force to $\vec{F}_{\text{on-ring}}$, $\vec{F}_{\text{on-string}} = -\vec{F}_{\text{on-ring}}$, which causes the kink in the string.



$F_z = F$ on ring by sting in z-direction

$$F_z = T - T\cos\theta \simeq T\frac{\theta^2}{2}$$

So the average upward force is $\overline{\theta^2/2} = E$

$$\overline{F}_z = mg\,\frac{\overline{\theta^2}}{2} = \frac{\overline{E}}{2\overline{\ell}}$$

So the work done by string per time is

$$-\frac{dE}{dt} = \overline{F}_z\,\frac{d\ell}{dt} = \frac{E}{2\ell}\,\frac{d\ell}{dt}$$

Which implies (by multiplying by $\sqrt{l}$)

$$\frac{d}{dt}(E\sqrt{l}) = 0$$

As we will this was to be expected. For a slow change of a paramater, there is an <u>adiabatic invariant</u>. For the SHO this evaluates to the energy per frequency

$$\boxed{\frac{E(t)}{\omega(t)} \simeq \text{constant}}$$

<u>General Discussion:</u>

- Consider a 1D system with 1D motion that is periodic and subject to a external parameter $\lambda(t)$ wich is slow

$T \equiv$ period of oscillation $\longrightarrow T \dfrac{d\lambda}{dt} \ll \lambda$ ⟵ λ is e.g. the length of string $l(t)$ changing in time.

- The Hamiltonian of the system $H(q,p;\lambda)$. The energy of the system is, $E = H(q,p,\lambda)$, which implicitly determines p, $p = p(q,E,\lambda)$.
- From Hamiltons EOM $dH/dt = \partial H/\partial t$

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \left(\frac{\partial H}{\partial \lambda}\right)_{p} \frac{d\lambda}{dt}$$

q will always be held fixed below

- Now we can average over a time scale Δt which is long compared to the period T, but short compared to the adiabatic time scale

$$\frac{\Delta E}{\Delta t} = \overline{\left(\frac{\partial H}{\partial \lambda}\right)_p} \frac{\Delta \lambda}{\Delta t}$$

- Where the average means averaged over a cycle

$$\overline{\left(\frac{\partial H}{\partial \lambda}\right)}_{qp} = \frac{1}{T} \oint \left(\frac{\partial H}{\partial \lambda}\right)_p dt \leftarrow$$

This is the time averaged force exerted by the string on the ring in our example

$$T \equiv \oint dt \equiv \text{period}$$

- rewriting $dt = dq/\dot{q} = dq/(\partial H/\partial p)_\lambda$

$$\frac{\Delta E}{\Delta t} = \frac{\Delta \lambda}{\Delta t} \frac{\oint \left(\frac{\partial H}{\partial \lambda}\right)_p \frac{dq}{\left(\frac{\partial H}{\partial p}\right)_\lambda}}{\oint \frac{dq}{(\partial H/\partial p)_\lambda}}$$

This trajectory is over a cycle with fixed λ where E is an independent variable and constant variable. p is regarded as a function of the independent variables E, λ through the implicit equation $E = H(p(E,\lambda),\lambda)$

- Now at fixed λ, the energy is constant and is an independent var.

$$\left(\frac{\partial E}{\partial \lambda}\right)_E = 0 = \left(\frac{\partial H}{\partial \lambda}\right)_p \cdot 1 + \left(\frac{\partial H}{\partial p}\right)_\lambda \left(\frac{dp}{d\lambda}\right)_E \qquad p = p(q, E, \lambda)$$

$$\text{or} \quad (\partial H/\partial \lambda)_p / (\partial H/\partial p)_\lambda = -(\partial p/\partial \lambda)_E$$

So

$$\frac{\overline{\Delta E}}{\Delta t} = \frac{\Delta \lambda}{\Delta t} \frac{\oint -(\partial p/\partial \lambda)_E \, dq}{\oint (\partial p/\partial E)_q \, dq}$$

Re-arranging

$$\oint \underbrace{\left(\frac{\partial p}{\partial E}\right) \frac{\Delta E}{\Delta t} + \frac{\partial p}{\partial \lambda} \frac{\Delta \lambda}{\Delta t}}_{\Delta p/\Delta t} \, dq = 0$$

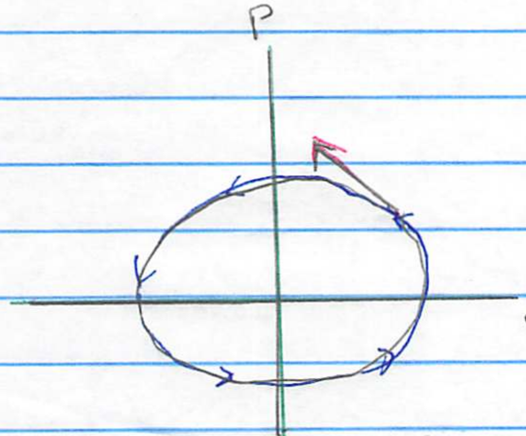
And thus

$$\boxed{\frac{\Delta I}{\Delta t} = 0 \qquad I = \oint p \frac{dq}{2\pi}}$$

inserting 2π is conventional in the classical theory

During the evolution of the system with a time dependent parameter λ, the Energy will change in time but the adiabatic invariant $I(E,\lambda) \equiv \oint p \, dq/2\pi$ will remain fixed in time.

- To understand what I means, first note that we may use the curl theorem:



$\vec{V} = (p, 0)$ $\quad d\vec{l} = (dq, dp)$

Then

$$\oint \vec{V} \cdot d\vec{l} = \int (\nabla \times \vec{V}) \cdot d\vec{a}$$

To write $\oint_{\text{line}} p\,dq = \int_{\text{area}} dp\,dq$

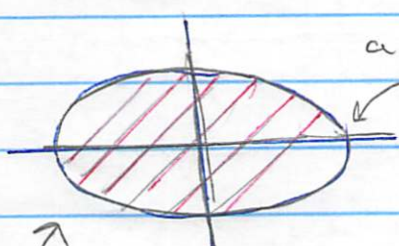
$$\boxed{I = \int_{\substack{\text{area} \\ \text{determined by } E}} \frac{dq\,dp}{2\pi}}$$

So the phase space area remains fixed in time

- For a SHO:

$$E = \frac{p^2}{2} + \frac{\omega^2 q^2}{2}$$

$b = \sqrt{2E}$



$a = \sqrt{\dfrac{2E}{\omega^2}}$

$\ominus$

Area of Elipse

$A = \pi a b$

$$I = \oint p\,dq \,/ 2\pi$$

$$= \int_{\text{area}} \frac{dp\,dq}{2\pi}$$

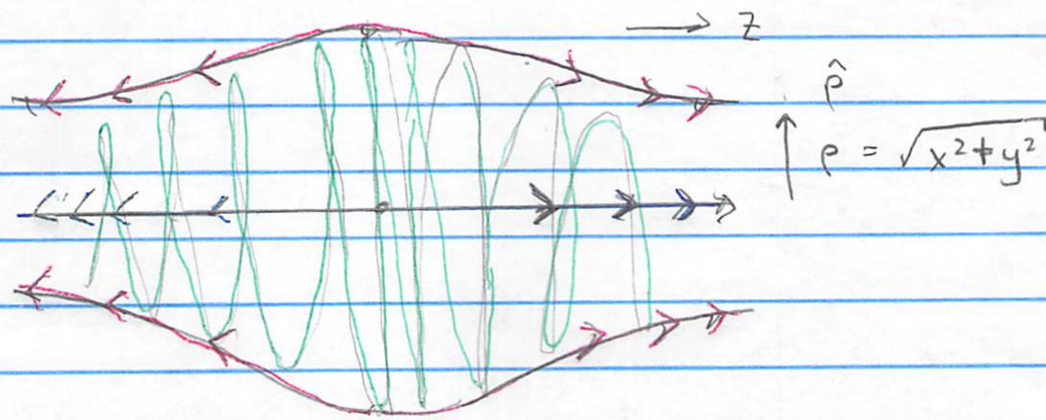
$$I = \frac{E}{\omega}$$

- So the energy per frequency is constant!

Example - Magnetic Confinement of Charged Particle

- Consider a fast moving electron in a magnetic field which grows slowly along the z-direction, e.g. $B_z(z) = B_0 (1 + z^2/a^2)$

- Since $\nabla \cdot B = 0$ there is a small correction to $\vec{B} = B_z(z)\hat{z} - \frac{1}{2} B'(z) \rho \hat{\rho}$ in the directions perpendicular to the z-axis



- As the charged particle flys toward the region of high field, the transvese (x,y) kinetic energy increases, and the particle's longitudinal kinetic energy decreases until it reaches a stopping point.

Analysis

- $L = \frac{1}{2} m v^2 + \frac{e}{c}\vec{v} \cdot \vec{A}$

- Now for a constant magnetic field in the z-direction

$$\vec{A} = \frac{B_0}{2}(-y, x, 0) + \text{gradent corrections if } B \text{ is not constant}$$

- The conserved energy (Hamiltonian function)

$$h(q, \dot{q}) = \frac{\partial L}{\partial \vec{v}} \cdot \vec{v} - L = \frac{1}{2} m v^2 = E$$

i.e for $\vec{v} = (\vec{v}_\perp, \dot{z})$

$$E = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} m v_\perp^2$$

The period of orbit is $2\pi/\omega_c$. Recall that for a particle in a magnetic field the "cyclotron" frequency is $\omega_c = eB/mc$. and $\frac{1}{2} m v_\perp^2 = \frac{1}{2} m (\omega_c R)^2$

- Now the particle has small v_z. We evaluate the adiabatic invariant for $v_z = 0$, and then recognize that if z and v_z change, the adiabatic invariant will be fixed

$$I = \frac{1}{2\pi} \oint_{\text{for circular orbit}} p \cdot dq$$

- Now for a circular orbit

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + \frac{e}{c}\cdot\vec{A}$$

- Now algebra determines the integral invariant for $\dot{z} = 0$

$$I = \frac{1}{2\pi}\oint \underbrace{\left(m\vec{v} + \frac{e}{c}\cdot\vec{A}\right)}_{\vec{p}} \cdot \underbrace{\vec{v}\, dt}_{d\vec{q}}$$

Use the circulation thrm and $\vec{B} = \nabla\times\vec{A}$

$$\Phi = \oint \vec{A}\cdot d\vec{r} = \int \vec{B}\cdot d\vec{a}$$

$$= -B\pi R^2$$

$$I = \frac{1}{2\pi}\oint m v_\perp^2\, dt + \frac{e}{2\pi c}\underbrace{\oint \vec{A}\cdot d\vec{r}}_{\Phi_0 = -B\pi R^2}$$

$$= \frac{1}{2\pi} m v_\perp^2 \frac{2\pi}{\omega_c} - \frac{e}{2\pi c} B\pi R^2$$

Use $\omega_c = eB/mc$

$$\frac{1}{2} m v_\perp^2 = \frac{1}{2} m(\omega_c R)^2$$

$$I = \frac{1}{2} m v_\perp^2 \left(\frac{mc}{eB}\right)$$

so

$$\boxed{\frac{1}{2} m v_\perp^2 = I\,\omega_c \qquad \omega_c = \frac{eB}{mc}}$$

- Now we can use that I is approximately constant as z - slowly changes

$$\frac{1}{2} m \dot{z}^2 + \frac{1}{2} m v_\perp^2 = E$$

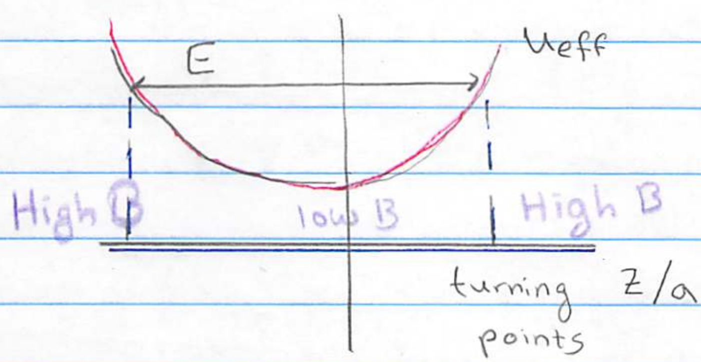
$$\frac{1}{2} m \dot{z}^2 + I \frac{e B(z)}{mc} = E$$

- So for $B(z) = B_0 (1 + z^2/a^2)$ we can describe the motion as that of an effective potential

$$\frac{1}{2} m \dot{z}^2 = E - U_{eff}(z)$$

where

$$U_{eff} = \frac{I e B_0}{mc} \left(1 + \frac{z^2}{a^2} \right)$$



which can be used to evaluate the period of oscillations and the turning points