Symplectic Structure and Canonical Transformation

- Recall that most of the "cool" features of flow in phase space follow from the structure of Hamilton's Equations:

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

- In particular, the phase space volume is constant:

$$d(\Delta P\, \Delta Q) = \left(\frac{\partial \dot{Q}}{\partial q} + \frac{\partial \dot{P}}{\partial p}\right) dt\, \Delta p\, \Delta q = 0$$

since $\frac{\partial^2 H}{\partial p \partial q} = \frac{\partial^2 H}{\partial q \partial p}$

- We will consider the most general change of coordinates that preserves the form of Hamilton's equations. The map in phase space is

$$q \to Q(q,p) \qquad p \to P(q,p)$$

- It is helpful to have a unified notation:

$$\vec{z} \equiv \begin{pmatrix} q^1 \\ \vdots \\ q^n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \quad 2n \text{ dimensional vector}$$

Then Hamilton's equations can be written

$$\boxed{\frac{dz^i}{dt} = J^{ij}\frac{\partial H}{\partial z^j}} \quad \text{with} \quad J^{ij} \equiv \underbrace{\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}}_{2n \times 2n \text{ matrix}}$$

Hamilton's Eqns

"Symplectic matrix"

$n \times n$ identity matrix

- Then consider a change of variables

$$z^i \Longleftrightarrow y^i(z) \qquad H(z) = \tilde{H}(y(z)) \equiv \tilde{H}(y)$$

new Hamiltonian

- The jacobian of the transformation is

$$M^i_{\ j} \equiv \partial y^i / \partial z^j \longleftarrow \text{Jacobian of transformation}$$

And the equations for y^i read:

$$\frac{dy^i}{dt} = \frac{\partial y^i}{\partial z^j}\dot{z}^j = \frac{\partial y^i}{\partial z^j} J^{jl}\underbrace{\left(\frac{\partial \tilde{H}}{\partial y^k}\frac{\partial y^k}{\partial z^l}\right)}$$

$$= (M J M^T)^{ik}\frac{\partial \tilde{H}}{\partial y^k} \qquad \frac{\partial H}{\partial x} = \frac{\partial \tilde{H}}{\partial y}\frac{\partial y}{\partial x}$$

- In order for the transformation to be canonical we must have

$$\boxed{\frac{dy^i}{dt} = J^{ik}\frac{\partial \tilde{H}}{\partial y^k} \longleftarrow \text{same form as the } z\text{-equation}}$$

So a transformation is canonical iff

$$\boxed{M J M^T = J}$$

i.e, it preserves the symplectic matrix. Here

Infinitessimal Canonical Transformation

- Now we will determine the most general form of a canonical transformation of infinitessimal form

transformed q $\quad dQ/d\lambda \equiv \dot{Q}$

$$q^i \longrightarrow Q^i(\lambda) \simeq q^i + \dot{Q}^i(q,p)\,\lambda \qquad \lambda \ll 1$$

$$p_i \longrightarrow P_i(\lambda) \simeq p_i + \dot{P}_i(q,p)\,\lambda$$

transformed momentum $\quad \frac{dP}{d\lambda} \equiv \dot{P}(p,q)$

- Then

$$M \equiv \begin{pmatrix} \mathbb{1} + \lambda\,\partial\dot{Q}/\partial q & \lambda\,\partial\dot{Q}/\partial p \\ \lambda\,\partial\dot{P}/\partial q & \mathbb{1} + \lambda\,\partial\dot{P}/\partial p \end{pmatrix}$$

- Require that $M J M^T = J$, meaning that if we write

$$M = \mathbb{1} + \lambda\, m^{(1)} \qquad m^{(1)} \equiv \begin{pmatrix} \partial\dot{Q}/\partial q & \partial\dot{Q}/\partial p \\ \partial\dot{P}/\partial q & \partial\dot{P}/\partial p \end{pmatrix}$$

Then

$$m^{(1)} J + J\, m^{(1)T} = 0 \qquad J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

Or

$$\bigstar \quad \frac{\partial(\dot{P}_i)}{\partial(p_j)} + \frac{\partial(\dot{Q}^j)}{\partial q^i} = 0 \quad \longleftarrow \text{"curl free"} = D + A^T = 0$$

Note: $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} = \begin{pmatrix} -B + B^T & A + D^T \\ -D - A^T & C - C^T \end{pmatrix}$

In general such equations can be solved by introducing a new scalar function $G(q,p)$ such that

$$\boxed{\Delta \dot{Q}^i = \frac{\partial G(p,q)}{\partial p_i} \qquad \Delta \dot{P}_i = -\frac{\partial G(q,p)}{\partial q^i}}$$

One can always solve equations like ✶ in this way. Indeed, it is the same type of equation as one has when a vector field $\nabla \times \vec{E} = 0$ is curl free, and one can then write it as a gradient of a scalar, $\vec{E} = -\nabla \phi$

We say that G generates an infinitessimal transformation

Noether Theorem Revisited

- Consider an infinitessimal transformation generated by G. Then how is the Hamiltonian H changed by the transformation? Well...

$$\delta H = \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p = H(p+\delta p, q+\delta q) - H(p,q)$$

$$= \lambda\left(\frac{\partial H}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial G}{\partial q}\right)$$

$$\delta H = \lambda \{H, G\}$$

- Thus we see that if the canonical transformation leaves H invariant $\delta H = 0$ the generator $G(q,p)$ of the transform is constant in time, i.e. it is conserved

- Let us spell it out a bit more.
Under a canonical transformation (which does not explicitly depend on time)

$$\tilde{H}(P,Q) = H(p,q) \qquad \text{always true}$$

- But, usually $\tilde{H}(P,Q)$ has a different functional form. The transformation is a symmetry if $\tilde{H}$ has the <u>same</u> functional form:

$$\tilde{H}(P,Q) = H(P,Q) = H(p,q)$$

And in this case:

$$\delta H \equiv H(P,Q) - H(p,q) = 0$$

<u>Example:</u>

Take the Generator $G(p,q) = p$, the transformation generated by G is

$$q \rightarrow q + \frac{\partial G}{\partial p}\lambda \qquad p \rightarrow p - \frac{\partial G}{\partial q}\lambda$$

i.e.,

$$q \rightarrow q + \lambda \qquad p \rightarrow p$$

Cleary G leaves

$$h = \frac{p^2}{2m}$$

invariant so (the momentum) moment $G = p$ is conserved for this Hamiltonian