Hamiltons Equations from a variational procedure

- So far we derived Hamilton's Equations from the Lagrange equations of motion and the definition of the Hamiltonian. We can also derive the EOM from the action $H = p\dot{q} - L$ or $L = p\dot{q} - H$

$$S[q,p] = \int dt\, L = \int dt\, p\dot{q} - H(q,p)$$

- Now however we will independently vary p and q while keeping q fixed at ends (previously we kept q-fixed but $\dot{q}$ was varried at ends)

$$q \longrightarrow q + \delta q(t) \qquad \delta q(t_1) = \delta q(t_2) = 0$$

$$p \longrightarrow p + \delta p(t) \qquad p(t_1) + p(t_2) \text{ free to change}$$

OK then the action under the variation

$$S[q+\delta q, p+\delta p] = \int dt \, (p+\delta p)\left(\frac{d}{dt}(q+\delta q)\right)$$

So

$$- H(q+\delta q, p+\delta p)$$

$$\delta S = S[q+\delta q, p+\delta p] - S[q, p]$$

And

$$\delta S = \int \delta p \frac{dq}{dt} + \underbrace{p\frac{d\delta q}{dt}}_{\text{by parts}} - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p = 0$$

Integrating by parts:

$$\delta S = \cancelto{0}{p\,\delta q\Big|_{t_1}^{t_2}} + \int_{t_1}^{t_2} dt \left(\frac{dq}{dt} - \frac{\partial H}{\partial p}\right)\delta p - \left(\frac{dp}{dt} + \frac{\partial H}{\partial p}\right)\delta q$$

The endpoints vanish sence $\delta q(t_1) = \delta q(t_2) = 0$,
Leading to (since δp and δq are arbitrary)

$$\boxed{\begin{aligned} \frac{dq}{dt} &= \frac{\partial H}{\partial p} \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \end{aligned}}$$

Hamilton Equations