The Hamilton Jacobi Equation

- Take the onshell action

$$S(t\,q;\,Q)$$

- We know that

$$\left.\begin{aligned} &(1) \qquad \frac{\partial S}{\partial q} = p \\ &(2) \qquad \frac{\partial S}{\partial t} = -H \end{aligned}\right\} \qquad dS = p\,dq - H\,dt$$

- So we can combine (1) and (2) to find a PDE for S

$$\frac{\partial S}{\partial t} = -H\left(q, \frac{\partial S}{\partial q}\right)$$

Which can be solved (we will do so below)

- The importance of the Hamilton-Jacobi theory for classical mechanics stems from its connection to quantum mechanics

★ As we will see $S(t,x)$ is the phase of the wave-fcn in the semi-classical limit

- Compare to the WKB approximation of the Schrödinger equation:

$$\frac{d\psi}{dt} = -\frac{i}{\hbar} H(x, \hat{p})\psi$$

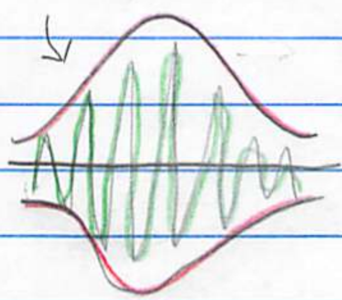
with

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

- Then write

$$\psi = \underbrace{A(t,x)}_{\text{slow, approx constant}}\ \underbrace{e^{\overbrace{iS(t,x)}^{\text{slow}}/\hbar}}_{\text{fast phase}} \quad \text{small}$$

- Typical wave fcn in the classical limit looks like this, and thus:



$$\frac{d\psi}{dt} = \frac{d}{dt} A e^{iS(t,x)/\hbar}$$

$$\simeq A\, e^{iS(t,x)/\hbar}\left(\frac{\partial S}{\partial t}\,\frac{i}{\hbar}\right) + O(\hbar)$$

- Similarly:

$$\hat{p}\psi = -i\hbar\frac{\partial\psi}{\partial x} \simeq A e^{iS/\hbar}\left(\frac{\partial S}{\partial x}\right) + O(\hbar)$$

$$\hat{p}^2\psi = -i\hbar\frac{\partial}{\partial x}(\hat{p}\psi) \simeq A e^{iS/\hbar}\left(\frac{\partial S}{\partial x}\right)^2 + O(\hbar)$$

So the Schrödinger Equation becomes

$$A e^{iS/\hbar} \frac{i}{\hbar}\frac{\partial S}{\partial t} = \frac{-i}{\hbar} H(x, \partial S/\partial x) \; A e^{iS/\hbar}$$

i.e.

$$\frac{\partial S}{\partial t} = - H(x, \partial S/\partial x)$$

the Hamilton Jacobi equation, S is the phase of Ψ!

The Hamilton Jacobi Equation Example

- Consider solving the Hamilton - Jacobi Equation for $U = -Fx$ a constant force

$$H = \frac{p^2}{2m} + U(x) \qquad U(x) \equiv -Fx$$

- So we need to solve

$$\frac{\partial S}{\partial t} = \frac{-}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + U(x)$$

- The only real way to solve the Hamilton - Jacobi equation, is the following;

$$S = S_t(t) + W(x) \qquad W(x) \; i$$

separated ansatz for S, i.e. a function of time $S(t)$ <u>plus</u> a function of space, $W(x)$

So substituting

$$\frac{\partial S_t}{\partial t} = -\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 - U(x)$$

- The RHS is only a function of t, while the LHS is only a function of x. Thus we require

(1) $$\frac{\partial S_t}{\partial t} = -E \quad \Rightarrow \quad S_t = -Et + S_0(E)$$

(2)
$$-\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 - U = -E$$

- While solving (2) gives:

$$\frac{\partial W}{\partial x} = \pm\left(2m(E-U(x))\right)^{1/2}$$

(☆) $$W(x) = \pm\int^x dx\,\left(2m\left(E-U(x)\right)\right)^{1/2}$$ using $U = -F_0 x$

One could integrate this but it is not instructive as ☆

Solution to Hamilton - Jacobi Eqn for 1D motion

So

$$\boxed{S(t x; E) = -Et + \int^{x} (2m(E - U(x)))^{1/2}\, dx + \delta(E)}$$

- In the WKB theory one has similarly

$$\psi \propto e^{iS/\hbar} \propto e^{-iEt + \int^{x} p\, dx + \delta(E)}$$

Where

$$\frac{p^2(x)}{2m} = E - U(x) \implies p(x) = \sqrt{2m(E-U)}$$

- The classical trajectories are points of stationary phase. For most paths there will be a destructive interference as the different parts of the add together with random phases;

$$\frac{\partial S}{\partial E} = 0$$

← stationary phase condition !!

$$= -t + \int^{x} \sqrt{\frac{m}{2}} \frac{1}{(E - U(x))^{1/2}} + \underbrace{\delta'(E)}_{\equiv t_0} = 0$$

Or

$$\int^{x} \sqrt{\frac{m}{2}} \frac{dx}{(E - U(x))^{1/2}} = t - t_0$$

This should be familiar. For 1D motion:

$$\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + U(x) = E$$

So

$$\sqrt{\frac{m}{2}} \frac{dx}{dt} = \pm (E - U(x))^{1/2}$$

And thus

$$\boxed{\pm \int_{x_0}^{x} \sqrt{\frac{m}{2}} \frac{dx}{(E - U(x))^{1/2}} = t - t_0}$$

Aside:

- In general we do not need to appeal to quantum mechanics (though I find it easiest, and you will too!)

The solution to the Hamilton-Jacobi equation will take the following form:

$$S(t q; P) \quad \text{the energy } E \text{ in our example}$$

Where P is a generalized momentum. It can be regarded as a generator of canonical trans-form

$$\Phi(t, q, P)$$

of the second type.

- Then the new coordinates and momenta are

$$p = \frac{\partial S}{\partial q} = \frac{\partial \Phi}{\partial q}(t, q, P)$$

$$Q = \frac{\partial S}{\partial P} = \frac{\partial \Phi}{\partial P}(t, q, P) \quad \text{with}$$

$$\underline{H} = h + \frac{\partial S}{\partial t} = h - h = 0 \quad \leftarrow \text{the new hamiltonian is zero!}$$

- Thus, since the transformed Hamiltonian is zero

$$\dot{Q} = \frac{\partial H}{\partial P} = 0 \quad \text{and thus} \quad Q = \beta, \ \text{constant},$$

which could be set to zero.

- Thus, for a trajectory satisfying the eom we have

$$\beta = \frac{\partial S}{\partial P}(t, q(t), P), \quad \text{in our example } P = E \text{ and}$$

$$\beta = -t + \int^{x} \sqrt{\frac{x}{2}} \frac{dx}{(E - U(x))^{1/2}} + \underbrace{\delta'(E)}_{t_0}$$

- So β is just adding to t_0, giving a new $\tilde{t}_0 = t_0 - \beta$

leading to

$$t - \tilde{t}_0 = \int_{x_0}^{x} \sqrt{\frac{m}{2}} \frac{1}{(E - U(x))^{1/2}}$$

as before