The Kepler Problem and HJ theory

- Now let us give one more example of the HJ theory now in two dimensions

$$L = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)$$

$$U(r) = -k/r$$

- The corresponding Hamiltonian is:

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r)$$

- with HJ equation

$$\frac{\partial S}{\partial t} = -H\left(r, \theta, \frac{\partial S}{\partial r}, \frac{\partial S}{\partial \theta}\right) \qquad ★★$$

- Then we will separate variables as before

$$S = -\underline{\underline{E}}\,t + W(r, \theta) \qquad \text{yielding:}$$

$$E = \frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + U(r)$$

★ E is a constant here, which is interpreted as energy from ★★

- Now given the form of the theory we separate variables further

$$W(r,\theta) = W_r(r) + W_\theta(\theta)$$

- This yields

$$\underbrace{E}_{\text{const}} = \underbrace{\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + U(r)}_{\text{fcn of } r} + \underbrace{\frac{1}{2mr^2}}_{\text{fcn of } r}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2$$

- So following the usual logic of separation of variables, since we could leave r fixed and change θ, and all fcns of r are constant, so we must have

$$\left(\frac{\partial W_\theta}{\partial \theta}\right) = \text{const} \equiv \ell \qquad W_\theta = \ell\,\theta$$

will interpret as angular momentum later

- So rearranging

$$E = \frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + \underbrace{\frac{\ell^2}{2mr^2} + U(r)}_{U_{eff}(r)}$$

- And we can rearrange this

$$W_r(r) = \pm\int^r dr' \left(2m\left(E - U_{eff}(r')\right)\right)^{1/2}$$

- So the complete solution is

$$S = -Et + \ell\theta + \int^r dr' \left(2m\left(E - U_{eff}(r)\right)\right)^{1/2}$$

- Now we have the equations of canonical transform

makes sense

$$\frac{p_r^2}{2m} + U_{eff} = E$$

$$p_r = \frac{\partial S}{\partial r} = \sqrt{2m\left(E - U_{eff}(r)\right)}$$

$$p_\theta = \frac{\partial S}{\partial \theta} = \ell$$ ← this justifies calling ℓ the angular momentum

momenta conjugate to r, θ

- Then the stationary phase conditions give the equation of motion for r vs. t and θ vs. t

$$\frac{\partial S}{\partial E} = -t + \int^r dr' \sqrt{\frac{m}{2}} \frac{1}{\left(E - U_{eff}(r)\right)^{1/2}} = 0$$

stationary phase, this is $r(t)$ for a particle in a 1D potential well

- Finally using $E - U_{eff}(r) = E + k/r - \ell^2/2mr^2$ we find

this is the Ellipse

$$\frac{\partial S}{\partial \ell} = \theta - \int^r dr \frac{\ell^2/r^2}{\left(2m\left(E + k/r - \ell^2/2mr^2\right)\right)^{1/2}} = 0$$

look back at Kepler notes!

- this gives after changing variables to $u \equiv 1/r$, and completing the square and doing the integral over u (see kepler notes!)

$$\theta = \cos^{-1}\left(\frac{1/r - 1}{\sqrt{1 + E/\mathcal{E}_0}}\right) \qquad \mathcal{E}_0 \equiv \frac{mk^2}{2\ell^2}$$

or

$$\frac{1}{r} = 1 + e\cos\theta \qquad \text{with} \quad e = (1 + 2E\ell^2/mk^2)^{1/2}$$

So we see how simply we get the elliplic motion!