The Onshell Action

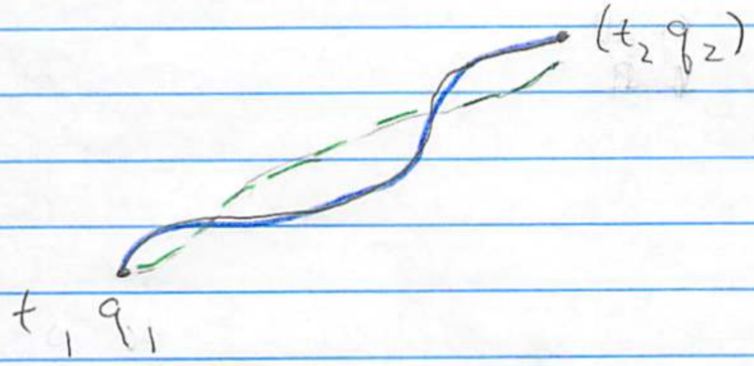
- So far we considered

$$S[q(t)] = \int_{t_1}^{t_2} L \, dt'$$

or

$$S[p,q] = \int_{t_1}^{t_2} p\,dq - H(q,p)\,dt'$$

- Then we considered a variation with the endpoints fixed finding the trajectory which extremizes the action.

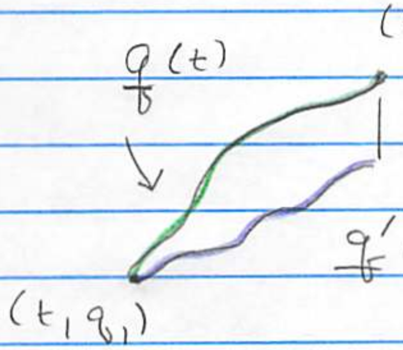


- If we substitute $q(t; t_1 q_1, t_2 q_2)$ into the action we get a number which depends on the endpoints

$$\boxed{\underline{S}(t_2 q_2; t_1 q_1) = S[q]}$$

As we will see $\underline{S}(t_2 q_2; t_1 q_1)$ is of great physical significance

How Does $\underline{S}$ depend on t_2 and q_2



$\Delta q = \delta q(t_2)$

Δq is the Change in the value of q_2

$q'(t) = q(t) + \delta q(t)$

new path Satisfying EOM ⓦ different b.c

$$\Delta \underline{S} = S[q + \delta q(t)] - S[q]$$

$$= p\,\delta q\Big|_{t_1}^{t_2} - \int dt \left(-\frac{d}{dt}\frac{\partial L}{\partial \dot q} + \frac{\partial L}{\partial q} \right) \delta q$$

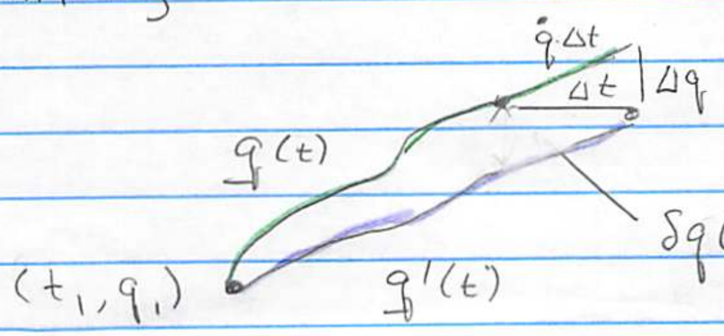
$$= p\,\delta q(t_2)$$

$$\Delta \underline{S} = p\,\Delta q$$

i.e.

$$\boxed{\frac{\partial \underline{S}(t_2\, q_2)}{\partial q_2} = p(t_2)}$$

Similarly if we displace in time, but leave the coordinate q unchanged, $q(t_2 + \Delta t) = q(t_2)$



$\delta q(t_2) = -\dot q\, \Delta t$

- Then if $q(t)$ follows the classical path

$$\Delta \underline{S} = L\,\Delta t$$

$$= \frac{\partial S}{\partial t_2}\Delta t + \frac{\partial S}{\partial q_2}\Delta q$$

- So dividing by Δt

$$\frac{\partial S}{\partial t_2} = L - \underbrace{\left(\frac{\partial S}{\partial q_2}\right)}_{p(t_2)}\frac{\Delta q}{\Delta t}$$

$$= L - p_2\dot{q}_2$$

$$= -h(p,q)$$

Dependence of Onshell action on endpoints

- So summarizing

$$\frac{\partial S}{\partial t_2} = -E \qquad \frac{\partial S}{\partial q_2} = p_2$$

$$\boxed{d\underline{S} = p_2\,dq_2 - h(p,q)\,dt_2}$$

In general, we kept the initial end point $(t_1 q_1)$. If these are also changed then we have analogously

$$d\underline{S}(t_2 q_2; t_1 q_1) = p_2\,dq_2 - p_1\,dq_1 - h_2\,dt_2 + h_1\,dt_1$$

The Onshell Action as Generator of Canonical Transform

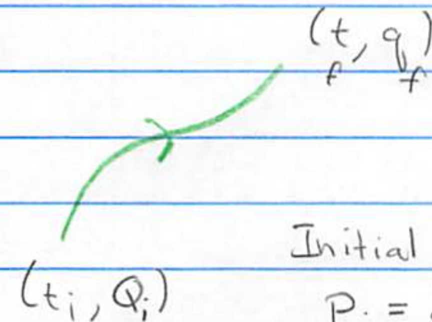
- Write

final coord

$$\underline{S}(t_f\, q_f;\; t_i\, Q_i)$$

final time

initial coord



Initial Momentum
$$P_i = \frac{\partial \underline{S}}{\partial Q_i}$$

- This can be used to generate a canonical transform

$$d\underline{S} = p_f\, dq_f - P_i\, dQ_i \;-\; \underline{h}\, dt$$

Compare

$$dF = p\, dq - P\, dQ \;-\; (h - \underline{H})\, dt$$

- Thus the onshell $\underline{S}$, generates a canonical map which takes the final p_f, q_f back to the initial one

$$p_f = \partial \underline{S} / \partial q_f$$

$$-P_i = \partial \underline{S} / \partial Q_i$$

New Hamiltonian is zero

$$\underline{H} = h + \partial S/\partial t = h - h = 0 \;!!$$

- Thus in the new coordinate system the new Hamiltonian is trivial, and knowledge of $\underline{S}$ maps the time dependendent problem onto a stationary one, essentially solving the problem. So now we "only" need to find $\underline{S}$