Problem 1. 2d isotropic oscillator

Consider the 2d harmonic oscillator which is isotropic

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + (\omega_0 x_1)^2 + (\omega_0 x_2)^2\right) \tag{1}$$

This is an example of an integrable system, which means if the phase space consists of $2n$ generalized coordinates there are $2n - 1$ constants of the motion. We will find and interpret these constants here.

(a) Show that

$$J_3(\boldsymbol{r}, \boldsymbol{p}) = \frac{1}{2}\left(x_1 p_2 - p_1 x_2\right) \tag{2}$$

generates rotations in the plane. Why is it constant in time?

(b) Determine the infinitesimal transformation generated by

$$J_1(\boldsymbol{r}, \boldsymbol{p}) = \frac{1}{2\omega_0}\left(\frac{1}{2}p_1^2 + \frac{1}{2}\omega_0^2 x_1^2 - \frac{p_2^2}{2} - \frac{1}{2}\omega_0^2 x_2^2\right). \tag{3}$$

Show that the computed transformation leaves the Hamiltonian invariant, and that this implies that $\dot{J}_1 = \{J_1, H\} = 0$. Give a physical interpretation of this quantity.

(c) Use the Poisson theorem to deduce a third conserved quantity J_2:

$$J_2 = \frac{1}{2\omega_0}\left(p_1 p_2 + \omega_0^2 x_1 x_2\right) \tag{4}$$

Determine the associated infinitesimal canonical transformation generated by this conservation law, and verify that it is a symmetry of the Hamiltonian.

(d) We have found three integrals of motion. Using similar manipulations to part (c), one may show that

$$\{J_i, J_j\} = i\epsilon_{ijk} J_k\,, \tag{5}$$

and that

$$\left(\frac{H}{2\omega_0}\right)^2 = J_1^2 + J_2^2 + J_3^2 \tag{6}$$

Thus any random orbit is selected by choosing J_1, J_2, J_3 to lie on the surface of a sphere. Describe the motion of the orbit in each of the following limiting cases

(i) $J_1 = J_2 = 0$

(ii) $J_2 = J_3 = 0$

(iii) $J_1 = J_3 = 0$

Problem 2. Phase-space and its characteristic flow

(a) If the number of particles per phase space volume (called the phase-space density)

$$f(t,q,p) = \frac{dN}{d^n q d^n p} \tag{7}$$

is conserved, then the phase-space density obeys a conservation law

$$\frac{\partial f}{\partial t} + \frac{\partial (f\dot{q}^i)}{\partial q^i} + \frac{\partial (f\dot{p}_i)}{\partial p_i} = 0\,. \tag{8}$$

This equation of motion is analogous to a compressible fluid, where the density $\rho(t,\boldsymbol{x})$ satisfies the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0\,, \tag{9}$$

with $\boldsymbol{v}(t,\boldsymbol{x})$ the velocity of the fluid. Eq. (8) does not require Hamilton's EOM, it just says that once a particle always a particle, regardless of the EOM.

(i) Show that if Hamilton's EOM are also satisfied and particle number is conserved, the Liouville equation (also called the free-streaming Boltzmann equation)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i}\dot{q}^i + \frac{\partial f}{\partial p_i}\dot{p}_i = 0\,, \tag{10}$$

is satisfied, and that this equation can be written as

$$\partial_t f + \{f, H\}_{p,q} = 0\,, \tag{11}$$

(ii) Eq. (10) is analgous to an incompressible fluid, where $\nabla \cdot \boldsymbol{v} = 0$, and thus we have from Eq. (9)

$$\partial_t \rho + \boldsymbol{v} \cdot \nabla \rho = 0\,. \tag{12}$$

What is the phase-space analog of the incompressibility constraint $\nabla \cdot \boldsymbol{v} = 0$?

(b) Eqs. (10) and (11) imply that $f(t,q,p)$ that f is constant along the flow lines. Heuristically, this means that we can find the solution to the equation Eq. (11) by tracing the trajectories backward in time to the initial time t_0 where the initial condition $f_0(q,p)$ is specified. This is known as the method of characteristics, and we will develop this method here.

(i) Show by direct substitution that for a free particle $H = P^2/2m$ the solution to

$$\frac{\partial f(t,Q,P)}{\partial t} + \{f, H\}_{P,Q} = 0 \tag{13}$$

is

$$f(t,Q,P) = f_0(Q - \frac{P}{m}t, P)\,. \tag{14}$$

where $f_0(q,p)$ is the initial consition at time $t = 0$. The somewhat confusing minus sign is just a reflection of the familiar fact that if I want to translate a function $F(x)$ forward by a distance $\Delta x = vt$, I want the new function $F(x - vt)$.

(ii) Show more generally that the characteristic solution to Eq. (13) is

$$f(t, Q, P) = f_0(q(Q, P; t, t_0),\, p(Q, P; t, t_0))\,, \tag{15}$$

where $f_0(q, p)$ is the initial condition at time $t = t_0$.

Here the characteristic solution is as follows – start at time t_0 with q, p and flow forward in time to time t were the coordinates are Q, P. This flow determines the map $(q, p) \to Q(q, p; t, t_0)$ and $(q, p) \to P(q, p; t, t_0)$. The inverse map is $q(Q, P; t, t_0)$ and $p(Q, P; t, t_0)$. Thus the characteristic solution can be written or more loosely

$$f(t, Q, P) = f_0(q, p)\,. \tag{16}$$

Hint: To prove Eq. (15), first show that q, p obey the EOM

$$\partial_t q(Q, P; t, t_0) = -\left(\frac{\partial q}{\partial Q}\frac{\partial H}{\partial P} - \frac{\partial q}{\partial P}\frac{\partial H}{\partial Q}\right) \equiv -\{q, H\}_{P,Q} \tag{17}$$

$$\partial_t p(Q, P; t, t_0) = -\left(\frac{\partial p}{\partial Q}\frac{\partial H}{\partial P} - \frac{\partial p}{\partial P}\frac{\partial H}{\partial Q}\right) \equiv -\{p, H\}_{P,Q} \tag{18}$$

and then prove Eq. (15).

(iii) Using the same notation, what are

$$\partial_{t_0} q(Q, P; t, t_0) = ? \qquad \partial_{t_0} p(Q, P; t, t_0) = ? \tag{19}$$

(c) The phase space density at the initial time $t = 0$ is

$$f(0, x, p) = \frac{1}{2\pi \Delta x_0 \Delta p_0} \exp\left[-\frac{x^2}{2\Delta x_0^2} - \frac{(p - P_0)^2}{2\Delta p_0^2}\right] \tag{20}$$

(i) Determine the phase space distribution $f(t, x, p)$ at later time t for a group of free particles, i.e. $H(x, p) = p^2/2$.

(ii) Sketch contour in the phase-space (x, p) where $f(t, x, p)$ is $1/e$ of its maximum (with $e \simeq 2.718$), at time $t = 0$ and at a significantly later time.

For definiteness take units where $m = \Delta x_0 = \Delta p_0 = 1$ take $P_0 = 3\Delta p_0$.

(d) The phase space density at the initial time is

$$f(0, x, p) = \frac{1}{2\pi \Delta x_0 \Delta p_0} \exp\left[-\frac{(x - X_0)^2}{2\Delta x_0^2} - \frac{p^2}{2\Delta p_0^2}\right] \tag{21}$$

(i) Determine the phase space distribution $f(t, x, p)$ at later time t for a group of particles in a harmonic oscillator, i.e $H(x, p) = (p^2 + \omega_0^2 x^2)/2$.

(ii) Sketch contour in the phase-space (x, p) where $f(t, x, p)$ is $1/e$ of its maximum (with $e \simeq 2.718$) at time $t = 0$ and at several subsequent times.

For definiteness take units where $m = \Delta x_0 = \Delta p_0 = 1$. Take $X_0 = 3\Delta x_0$ and $m\omega_0 = 3\Delta p_0$