Problem 1. Equations of motion

(a) From the Euler-Lagrange equations, determine the partial differential equation of motion resulting from the following action:

$$S = \int dt\, dx \frac{1}{2}\mu(\partial_t q)^2 - \frac{T}{2}(\partial_x q)^2 - V(q)$$

where $V(q) = \lambda q^4$. Also determine the canonical stress tensor for this action.

(b) Determine the partial differential equation of motion resulting from the following action:

$$S = \int dt\, dx \frac{1}{2}\mu(\partial_t q)^2 - \frac{T}{2}(\partial_x q)^2 - \alpha q \partial_x^4 q$$

(c) Determine the partial differential equation of motion resulting from the following action:

$$S = \int dt\, dx \frac{1}{2}\mu(\partial_t q)^2 - \frac{T}{2}(\partial_x q)^2 - \alpha(\partial_x^2 q)^2$$

Compare to part (b) and comment on similarities and differences.

Problem 2. Group velocity

(a) (Optional – Warm up) This is lecture material, you can just copy the derivation. Find the eigen-frequencies and normal modes of N particles which are connected by springs and which can move along a circle, $i.e$ periodic boundary conditions $q_{N/2}(t) = q_{-N/2}(t)$. All particles have mass m and are separated by a distance a; the spring constants of all of the springs are the same and equal γ.

Express a general real solution $q_\ell(t)$ with $\ell = 1 \ldots N$ as a linear superposition of the eigen-modes.

(b) Determine the rate that work is done by the $j-1$ mass on the j-th for any general real solution, $q_\ell(t)$ with $\ell = 1 \ldots N$.

(c) Show that if the motion is that of a wave traveling along the circle $\propto e^{ikx}$, the time averaged energy flux (i.e. the time average of the work in (b)) equals the product of the (time averaged) energy density and the group velocity of the system.

For computing the averages it might be useful to note that if $A(t) = \mathrm{Re}[A_\omega e^{-i\omega t}]$ and $B(t) = \mathrm{Re}[B_\omega e^{-i\omega t}]$ then

$$\overline{A(t)B(t)} = \frac{1}{4}\left(A_\omega B_\omega^* + A_\omega^* B_\omega\right) \tag{1}$$

Prove this result for yourself. It is quite helpful if simple.

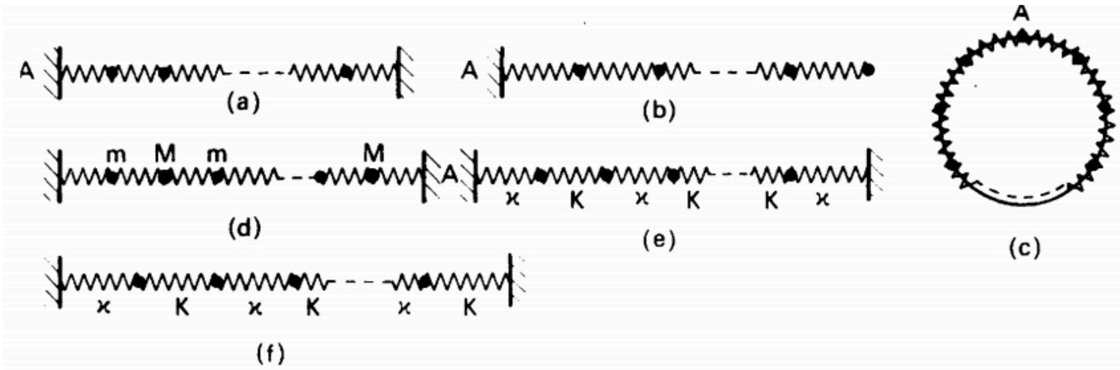

FIG. 33

Figure 1: A number of figures for the problems on springs

Problem 3. Normal modes and standing waves

In class we considered periodic boundary conditions. Here you will work out the modifications to the eigenfrequencies and functions with different boundary conditions.

(a) Determine the frequencies and eigenmodes of the eigen-vibrations for a system of N identical particles with masses m connected by identical springs with elastic constants γ and spatial separation a. Take the end points of the chain to be fixed (Fig. 33a).

You should find that general solution is a superposition of normal modes $\propto \sin(kx)$ labelled by an integer m

$$q_j(t) = \sum_{m=1}^{N} \mathcal{A}_m \cos(-\omega(k_m)t + \varphi_m) \sin(k_m j a) \tag{2}$$

with

$$k_m = \frac{\pi}{a} \frac{m}{N+1} \qquad\qquad m = 1 \ldots N \tag{3}$$

where $\mathcal{A}_m$ and φ_m are determined by the initial conditions $q_j(0)$ and $\dot{q}_j(0)$.

(b) Repeat (a) when only one end is fixed, and the other end may vibrate freely (Fig. 33b).

You should find that the general solution is a superposition of normal modes

$$q_j(t) = \sum_{m=0}^{N-1} \mathcal{A}_m \cos(-\omega(k_m)t + \varphi_m) \sin(k_m j a) \tag{4}$$

with

$$k_m a = \pi \left(\frac{\frac{1}{2} + m}{N + \frac{1}{2}} \right) = \pi \left(\frac{1 + 2m}{2N+1} \right) \tag{5}$$

and

$$m = 0 \ldots N - 1 \tag{6}$$