Problem 1. Particle in an electro-magnetic field

A non-relativistic particle of charge q in a electro-magnetic field is described by the Lagrangian (try to remember this!)

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - q\phi + q\frac{\dot{\boldsymbol{r}}}{c}\cdot\boldsymbol{A} \tag{1}$$

where $\phi(t, \boldsymbol{r}(t))$ is the scalar potential, and $\boldsymbol{A}(t, \boldsymbol{r}(t))$ is the vector potential of electricity and magnetisim. The electric and mangic fields are related to ϕ and $\boldsymbol{A}$ through

$$\boldsymbol{E}(t, \boldsymbol{r}) = -\nabla\phi - \frac{1}{c}\partial_t\boldsymbol{A} \tag{2}$$

$$\boldsymbol{B}(t, \boldsymbol{r}) = \nabla\times\boldsymbol{A} \tag{3}$$

(a) Show that the Euler-Lagrange equations give the expected EOM for a particle experiencing the force law: $\boldsymbol{F} = q(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B})$.

(b) Compute the canonical momementa $\boldsymbol{p}$. How is this related to the so called kinetic momentum $\boldsymbol{p}_{\mathrm{kin}} = m\dot{\boldsymbol{r}}$? Does

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F} \tag{4}$$

with $\boldsymbol{p}$ the (canonical) momentum hold and $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}/c\times\boldsymbol{B})$? Explain.

(c) Determine the Hamiltonian $H(\boldsymbol{r}, \boldsymbol{p})$ and Hamiltonian function $h(\boldsymbol{r}, \dot{\boldsymbol{r}})$.

$H(\boldsymbol{r}, \boldsymbol{p})$ and $h(\boldsymbol{r}, \dot{\boldsymbol{r}})$ return the same value (at corresponding points), but have different functional forms (meaning that they have different dependences on the arguements). A mathematician would (correctly) say that they are different functions, but we (too) loosely say that they are the "same".

Problem 2. Legendre transform tutorial

The Legendre transform and its properties are used everywhere in all fields of physics. Recall that the Legendre transform of a concave up function $U(x)$ is[1]

$$V(s) \equiv \mathrm{max_x[sx - U(x)]} \tag{6}$$

where $\mathrm{max_x}$ denotes the maximum value of $sx - U(x)$ as x is changed. This maximum is a function of the external parameter s.

(a) Determine the Legendre transform of $\frac{1}{2}k(x-x_0)^2$

(b) (Optional. Not graded) Show that the Legendre transform of $\log(1-e^{-x})$ with $x > 0$ is

$$V(s) = -s \log s + (1+s)\log(1+s) \tag{7}$$

What is the appropriate range of s? Explain.

This problem is related to the entropy of an ideal gas of bosons.

(c) Consider a potential energy function $U(x)$ (concave up) for a particle in a potential. Suppose that an external force is now applied of magnitude f_0. Relate the minimum value of the new potential $U(x, f_0)$ energy function (in the presence of f_0) to the Legendre transform of U. How could you determine the minimum value of the potential energy $U(x)$ (without the force) from $V(s)$?. If you wish you can check your result using (a).

(d) Working in the setup of the previous problem, show that the x-coordinate of the minimum of $U(x, f_0)$ with and without the external force f_0 can be determined entirely from $V(s)$? Check your general expression using the result of (a).

(e) Show that

$$\frac{\partial^2 V(s)}{\partial s^2}\frac{\partial^2 U}{\partial x^2} = 1 \tag{8}$$

The Legendre transform of a potential with several coordinates, say x^1 and x^2 for example,

$$V(s_1, s_2) \equiv s_1 x^1 + s_2 x^2 - U(x^1, x^2) \tag{9}$$

In this case Eq. (8) becomes a matrix equation. Defining

$$V^{ij} \equiv \frac{\partial^2 V(s)}{\partial s_i \partial s_j} \tag{10}$$

$$U_{ij} \equiv \frac{\partial^2 U(s)}{\partial x^i \partial x^j} \tag{11}$$

[1] If the function where concave down one would look for a minimum instead of a maximum. So it probably would be better to define

$$V(s) = \mathrm{extrm}_x[sx - U(x)] \tag{5}$$

where extrm_x denotes the extremum of a function as x is changed. Sometimes $V(s)$ is defined with a relative minus to what is done here, e.g. $V(s) = \min_x[U(x) - sx]$. It may be (marginally more) helpful to adopt this definition in part (c) below.

the generalization of Eq. (8) reads[2]

$$V^{ij}U_{j\ell} = \delta^i_{\ \ell} \tag{12}$$

i.e. U and V are inverse matrices.

For the Lagrangian a discussed in class

$$L = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j + b_i\dot{q}^j - U(q) \tag{13}$$

Write down the Hamiltonian for this Lagrangian (derived in class) and describe how the theorem in Eq. (12), is corroborated by its form.

[2]The proof is a straightforward generalization of the 1d case.

Problem 3. A Routhian tutorial and the effective potential

Consider the Kepler Lagrangian again:

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - U(r) \tag{14}$$

There are two variables r and ϕ with associated momenta p_r and p_ϕ. The Hamiltonian is is formed by Legendre transforming with respect to r and ϕ

$$H = p_r\dot{r} + p_\phi\dot{\phi} - L(r, \dot{r}, \phi, \dot{\phi})\,. \tag{15}$$

It can be convenient to Legendre transform with respect to only some of the variables instead of all of them. We define the *Routhian*[3]:

$$R(r, \dot{r}, \phi, p_\phi) \equiv p_\phi\dot{\phi} - L(r, \dot{r}, \phi, \dot{\phi})\,, \tag{16}$$

which serves as a Hamiltonian for ϕ, but a Lagrangian for r. This is especially helpful when some of the coordinates are cyclic (ϕ in this case). The p_ϕ are then just constants (both in the equation of motion *and* in the action), and we have effectively a Lagrangian for the remaining (non-cyclic) coordinates.

(a) From the Lagrange equations of motion, show that the Routhian equations of motion (for a generic Lagrangian not just Eq. (14)) are

$$\frac{d\phi}{dt} = \frac{\partial R}{\partial p_\phi} \tag{17}$$

$$\frac{dp_\phi}{dt} = -\frac{\partial R}{\partial \phi} \tag{18}$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{r}} = \frac{\partial R}{\partial r} \tag{19}$$

(b) Determine $R(r, \dot{r}, \phi, p_\phi)$ for the Lagrangian in Eq. (14) and the Routhian equations of motion. You should find[4]

$$-R = \frac{1}{2}m\dot{r}^2 - V_{\text{eff}}(r, p_\phi) \tag{20}$$

where $V_{\text{eff}}(r, p_\phi)$ was defined in class and the equation of motions are

$$m\ddot{r} = -\frac{\partial V_{\text{eff}}(r, p_\phi)}{\partial r} \tag{21}$$

$$p_\phi = \text{const} \tag{22}$$

Now might be a good time to review the appropriate comments on bottom of pg.2 and 3 in lecture to appreciate the how the Routhian can help, i.e. we want $(\partial V_{\text{eff}}/\partial r)_{p_\phi}$.

[3] Edward John Routh was a physicist of some repute. He was also an outstanding educator at Cambridge.

[4] Note that the sign of R is conventional. The choice here is nice in that the Hamiltonian part of the equations (Eq. (17) and Eq. (18)) takes the form of Hamilton's equations. But then, R is minus the effective Lagrangian for the non-cyclic coordinates. We will get around this "difficulty" by presenting $-R$.

(c) A particle of mass m is confined to move on the surface of a sphere. It moves freely on the surface but experiences the acceleration of gravity g:

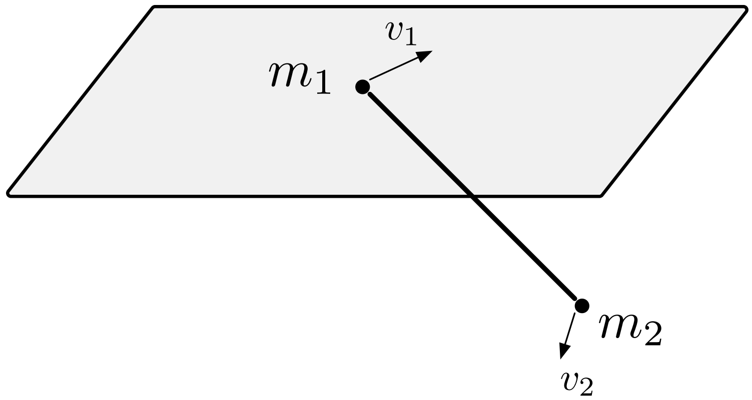
(i) Write down the Lagrangian for this system using the spherical angular variables θ, ϕ.

(ii) Form the Routhian for this system by Legendre transforming with respect to the cyclic coordinate.

(iii) Sketch the effective potential of θ for p_ϕ small and large, after defining what large and small means. Determine the stationary point of θ at large p_ϕ, and briefly describe the result physically.

Problem 4. A sliding conical pendulum

Consider two beads connected by a light rod of length ℓ. The first bead has mass m_1 and is constrained to lie in the x, y plane, but may move freely in this plane. The second bead has mass m_2 and can move freely in all three dimensions, and can pass freely through the x, y plane. The system sits in the earths gravitational field $\boldsymbol{g} = -g\,\hat{\mathbf{z}}$.



(a) Determine the distance from m_1 to the center of mass. You should find

$$\ell_{\rm cm} = \alpha\ell, \qquad \alpha \equiv \frac{m_2}{M}, \qquad M \equiv (m_1 + m_2)\,, \tag{23}$$

which establishes some notation used below.

(b) Clearly define some appropriate generalized coordinates for the system, and write down the Lagrangian of the system in terms of these coordinates.

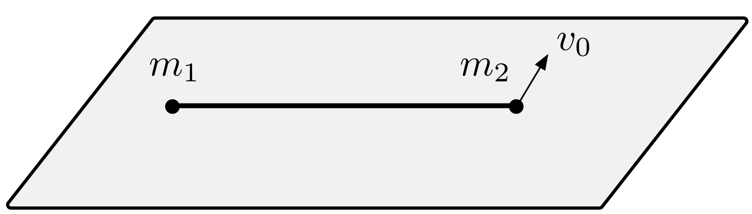
Hint: The cartesian coordinates (X, Y, Z) of the center of mass is an excellent choice. Then I used the the spherical coordinates θ and ϕ to orient the rod relative to the center of mass. I find the Lagrangian takes the form

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}m_0(\theta)\ell^2\dot{\theta}^2 + +\frac{1}{2}\mu\ell^2\sin^2\theta\dot{\phi}^2 + Mg\alpha\ell\cos\theta \tag{24}$$

where $m_0 = M\alpha^2\sin^2\theta + \mu$ and $\mu = m_1m_2/M$ is the reduced mass.

(c) Identify all integrals of the motion.

Now consider the case where the first bead is initially at rest and the second bead initially has velocity v_0 in the x, y plane, and perpendicular to the rod, before beginning to fall (see below).



(d) Describe qualitatively the subsequent motion of the system. In what Galilean frame is the motion periodic? Explain.

(e) (i) The pendulum swings down from an initial angle of $\pi/2$ relative to the vertical to a minimum angle. Determine this minimum angle.

You should find

$$\cos\theta_- = \frac{-1+\sqrt{1+4u^2}}{2u} \qquad\qquad \theta_- < \pi/2\,. \tag{25}$$

where $u = Mg\alpha\ell/\frac{1}{2}\mu v_0^2$.

(ii) Determine the associated period of the motion as a definite integral. Define what is meant by large and small v_0 and describe the motion qualitatively in these two limits.

You should show that this period takes the form

$$\mathcal{T} = \tau_0\, f(u, m_1/m_2) \tag{26}$$

where $\tau_0 \equiv \ell/v_0$ and $f(u, r)$ is a dimensionless function of u and the ratio of masses $r = m_1/m_2$. Use mathematica to plot to make a plot of $\mathcal{T}/(2\pi\tau_0)$ for $m_1 = m_2$.