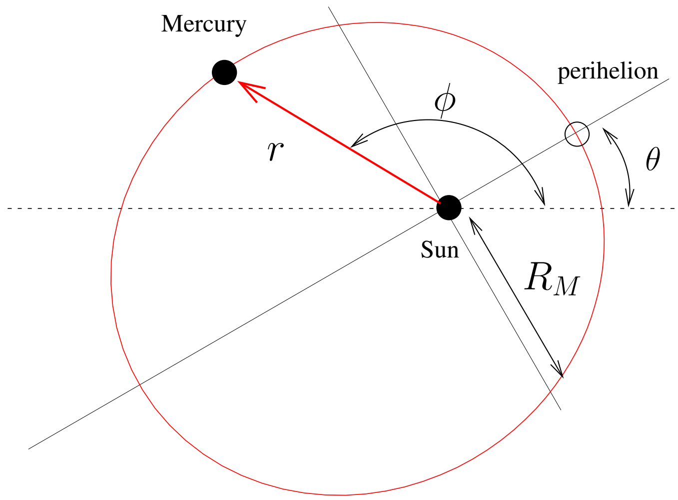
Problem 1. The precession of Mercury due to Jupiter

Recall that the trajectory of Mercury $r(\phi)$ is an ellipse with the sun at one focus as shown below. The perihelion (defined as the distance of closest approach) is rotated relative to the x-axis by an angle θ. The lattice rectum of Merucury is denoted R_M and is related to the angular momentum ℓ of the system (as discussed in class) but independent of the energy at fixed ℓ. The eccentricity of Mercury is small, $\epsilon = 0.2$, although it is the most eccentric of the Sun's planets.



Due to perturbations from the other planets, the angle of the perihelion θ changes (or processes) as function of time. The precession rate is very small. The contribution of Jupiter to the precession rate is of order 150 arcsec/century, or (since the orbital period of Mercury is 88 days) approximately 1.78×10^{-6} rad/turn.

The goal of this problem is to estimate Jupiter's contribution to the precession rate[1]. Specifically, we will model Jupiter as a thin ring of mass M_J at the orbital radius of Jupiter R_J, and compute how this ring perturbs Mercury's orbit and causes the perihelion of Mercury to precess. Jupiter's orbital radius is significantly larger than Mercury's, $R_J \simeq 10\, R_M$.

(a) Show that for $R_J \gg R_M$ the Lagrangian of Mercury interacting with the sun of mass $M_\odot$, and a ring of mass M_J and radius R_J is approximately

$$L \simeq \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{GmM_\odot}{r} + \alpha r^2\,, \tag{1}$$

and show the coefficient $\alpha = GmM_J/(4R_J^3)$.

Hint: Let the origin be at the center of the ring. Let $\boldsymbol{r}$ be the vector from the center of the ring to a point of interest (i.e. Mercury) close to the center. For simplicity assume

[1] Famously, general relativity also perturbs the classical orbit and contributes 43 arcsecs/century to the total precession rate. This "anomalous" precession of Mercury was measured in the nineteenth century by le Verrier and finally explained by Einstein in 1915. The total precession rate is approximately 550 arcsec/century

that $\boldsymbol{r}$ lies in the xy plane. Then we want to integrate the Newton gravitational potential $d\Phi = -GdM_J/|\boldsymbol{r} - \boldsymbol{R}_J|$ over the mass of the ring to determine the gravitational potential due to the ring at point $\boldsymbol{r}$.

To this end, let $\boldsymbol{R}_J$ is a vector from the center to a point on the ring. Show that for $r \ll R_J$

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{R}_J|} \simeq \frac{1}{R_J}\left(1 + \frac{r}{R_J}\cos(\phi) + \frac{r^2}{2R_J^2}\left(3\cos^2\phi - 1\right)\right) \tag{2}$$

where $\cos\phi$ is the angle between $\boldsymbol{r}$ and $\boldsymbol{R}_J$, and then integrate over ϕ.

(b) The orbit of mercury is characterized by its energy E and its angular momentum ℓ. Let us introduce some dimensionless variables to simplify the notation.

Since the unit of mass, time, and space are arbitrary we can (effectively) set three parameters to unity. Let us choose these three parameters to be ℓ, m, $k \equiv GM_\odot m$. Then all other scales are measured in these units. Thus, for instance, the only unit of length that can be constructed out of these three quantities can is

$$R_0 = \frac{\ell^2}{Gm^2 M_\odot}\,. \tag{3}$$

In class we showed that the lattice rectum of the unperturbed ellipse is $R_M = \ell^2/mk$, and thus the unperturbed lattice rectum $R_M = 1\,R_0$ is unity in these units. All distances will be measured in terms of R_0.

(i) What are the units of time T_0 and energy E_0 with this set of units?

(ii) Introduce a dimensionless radius $\underline{r} \equiv r/R_0$ and other suitable dimensionless variables, $\underline{t} \equiv t/T_0$ and $\underline{\alpha} \equiv \alpha/(E_0/R_0^2)$. Show that a dimensionless Lagrangian for the system is

$$\underline{L} = \frac{1}{2}\left(\frac{d\underline{r}}{d\underline{t}}\right)^2 + \frac{1}{2}\underline{r}^2\left(\frac{d\underline{\phi}}{d\underline{t}}\right)^2 + \frac{1}{\underline{r}} + \underline{\alpha}\,\underline{r}^2\,, \qquad \underline{L} = \frac{L}{E_0}\,, \tag{4}$$

where the dimensionless constant $\underline{\alpha}$ is of order

$$\underline{\alpha} \equiv \frac{M_J}{4M_\odot}\left(\frac{R_M}{R_J}\right)^3 \simeq 0.1 \times 10^{-6}\,. \tag{5}$$

Note that α has units energy per length squared, which is why its dimensionless version is scaled by E_0/R_0^2.

The dimensional analysis step amounts to setting $\ell = m = GM_\odot m = 1$ everywhere in the original Lagrangian. We actually gained a little something by this analysis, i.e. without doing any computation we learned that the effect of the perturbing ring is of order one part in 10^7.

To lighten the notation, stop underlining the variables in what follows.

(c) Determine the Hamiltonian of the dimensionless system, and use the Hamiltonian to determine the equations of motion. You should find

$$\dot{r} = p_r \,, \tag{6}$$

$$\dot{\phi} = \frac{p_\phi}{r^2} \,, \tag{7}$$

$$\dot{p}_r = \frac{p_\phi^2}{r^3} - \frac{1}{r^2} + 2\alpha r \,, \tag{8}$$

$$\dot{p}_\phi = 0 \tag{9}$$

You can set $p_\phi = \ell = 1$ to unity only after find the equations of motion. By rescaling our units we are setting the initial value for p_ϕ, i.e. ℓ, to 1, and then the equation of motion guarantee that it remains unity at all subsequent times.

For $\alpha = 0$. The minimum of the effective potential is at $r = 1$ and its minimum value is $E_{\rm min} = -1/2$. Recall from class that the eccentricity of the ellipse for $\alpha = 0$ in dimensionless units is

$$e = \sqrt{1 + E/|E_{\rm min}|} \tag{10}$$

Thus, for the real "Mercurial" orbit the energy difference $\epsilon \equiv E - E_{\rm min} = e^2|E_{\rm min}| \simeq 0.04\,|E_{\rm min}|$ is small, and the orbit is nearly circular up to small oscillations of around the minimum of the effective potential. We will use this almost circular approximation for $\alpha \neq 0$.

(d) Determine the radius and period for the circular orbit to first order in α. I find $r \simeq 1 + 2\alpha$

(e) Determine the period of radial oscillations for slight disturbances from this circular orbit to first order in α. I find

$$\tau_M \simeq 2\pi(1 + 7\alpha) \tag{11}$$

(f) Show that the angle of perihelion of the ellipse θ will advance by an angle $\Delta\theta = 6\pi\alpha$ (see picture above), every time the particle reaches the distance of closest approach.

Restoring units we find

$$\Delta\theta = 6\pi\underline{\alpha} = \frac{3\pi}{2}\frac{M_J}{M_\odot}\left(\frac{R_M}{R_J}\right)^3 \simeq 1.88 \times 10^{-6} \qquad \text{rad per turn}\,, \tag{12}$$

This should be compared to the experimental result of 1.78×10^{-6}rad/turn.

Problem 2. A scattering cross section

A particle of mass μ moves in the repulsive $1/r^2$ potential

$$U(r) = \frac{h}{r^2}\,, \quad h > 0\,.$$

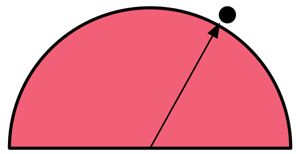
(a) Find equation for a generic trajectory $r(\phi)$ characterized with energy E and angular momentum $\ell \neq 0$. Follow the convention that the direction $\phi = 0$ points to the pericenter (point of closest approach).

(b) Find the time dependence on this trajectory, taking the time $t = 0$ at the pericenter.

(c) Find the differential scattering cross section $\frac{d\sigma(\theta)}{d\Omega}$ for a particle with energy E in this potential.

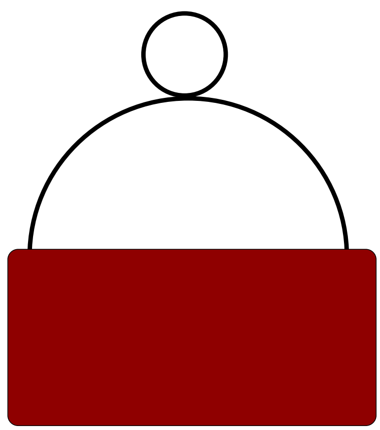
Problem 3. (Goldstein) A hoop on a cylinder

(a) First consider a small block of mass m on a cylinder of radius R on earth. If the block starts from rest on top of the cylinder, determine at what angle θ the block falls off the cylinder using the Lagrangian formalism to impose the constraint $r = R$.

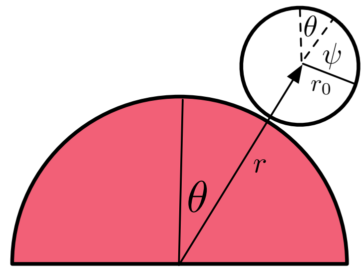


the coordinates are r, θ

(b) Now consider a uniform hoop of mass m and radius r_0 rolls without slipping on a fixed cylinder of radius R as shown in the figure. The only external force is that of gravity. If the cylinder starts rolling from rest on top of the bigger cylinder, use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder. You should find $\theta = 60^o$



(i) Setup some coordinates. I took those based on the picture below, but their are better choices. Determine the relaxation between the X and Y coordinates of a point on the rim of the hoop in terms of r, θ, ψ.



the coordinates are r, θ, ψ

(ii) Starting with the general expression

$$T = \frac{1}{2} \int dm v^2 \tag{13}$$

show that the kinetic energy is of the hoop is

$$T = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r_0^2 (\dot{\theta} + \dot{\psi})^2 \tag{14}$$

(iii) Determine a relation between $d\theta$ and $d\psi$ if the hoop rolls without slipping.

(iv) Introducing a Lagrange multiplier for r to enforce the constraint, and find the angle where the hoop falls off the cylinder. You should find $\theta_{\mathrm{fall-off}} = 60^o$

Problem 4. Constraints in the Hamiltonian Formalism

If the canonical variables p_i, q_i, t (with $i = 1 \dots N$) are not all independent but are related by auxiliary conditions of the form

$$\psi_k(p_i, q_i, t) = 0 \tag{15}$$

(with $k = 1 \dots m$) determine the modified Hamilton equations of motion by varying the appropriate action.

Problem 5. The first order formalism and the transition to the Hamiltonian

This problem uses the notion of Lagrange multipliers and Legendre transforms to understand the action in the Hamiltonian formalism. Previously we said that the action in the Hamiltonian formalism is

$$S[q,p] = \int dt \left(p\frac{dq}{dt} - H(p,q) \right) . \tag{16}$$

We showed that extremizing this action gives Hamilton's equations of motion and that these equations are equivalent to the Euler-Lagrange equations. We did not, however, derive $S[q,p]$ directly from the action of the Lagrangian, $S[q]$. We will do this in this problem.

The action principle says that the action is

$$S[q] = \int dt L(q,\dot{q}) \tag{17}$$

and the system will follow the trajectory $\underline{q}(t)$ which extremizes this action. Using a Lagrange multiplier called $p(t)$ (for reasons discussed below), we may separately vary the velocity $v(t)$ and $\dot{q}$ by defining

$$\hat{S}[q(t), v(t), p(t)] \equiv \int dt\, \hat{L}(q,\dot{q},v,p) \qquad\qquad \hat{L} \equiv L(q,v) - p(v-\dot{q})\,. \tag{18}$$

and require that $\delta\hat{S} = 0$ for independent variations of q, v, p. The Lagrange multiplier enforces that $v = \dot{q}$ at the level of the equations of motion rather than the action. This "theorist-gone-wild" procedure is known as the "first order formalism", and has been found to be useful in analyzing various rich theories (such as gravity) which have complicated constraints.

(a) Consider the Lagrangian

$$L = \frac{1}{2}m\dot{q}^2 - U(q) \tag{19}$$

Show that the equations motion following from $\delta\hat{S}[q,v,p] = 0$, reproduce Newton's laws. Does the Lagrange multiplier have an appropriate name? Explain.

(b) One way to to extremize $\hat{S}$ is to first extremize $\hat{S}$ with respect to p, v with q fixed, leaving a reduced action $S_{\rm red}[q]$ to be extremized later. Argue that this reduced action is the Lagrangian formulation $S[q]$ in Eq. (17).

(c) Now extremize $\hat{S}$ with respect to v first with q and p fixed, leaving a reduced action $S_{\rm red}[q,p]$ to be extremized later, and argue that this reduced action is the Hamiltonian formulation $S[q,p]$ in Eq. (16).