Problem 1. (Goldstein) A molecule with a right triangle

The equilibrium configuration of a molecule consists of three identical atoms of mass m at the vertices of a 45^o right triangle connected by springs of equal force constant k. The atoms are constrained to move in the xy plane. We will determine the modes of oscillation of this molecule.

Zero Modes:

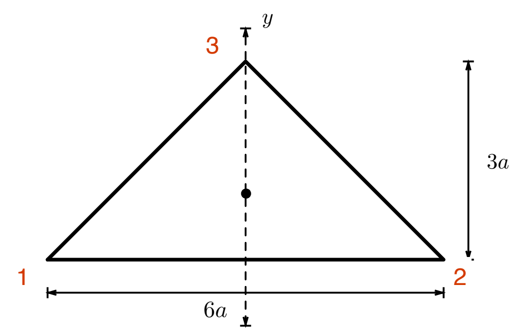
(a) The vectors in the space of displacements are labelled by

$$\vec{Q} = (x_1, y_1, x_2, y_2, x_3, y_3) \tag{1}$$

where (x_1, y_1) are the coordinates of particle 1, etc. Show that a displacement corresponding to a global rotation parameterized by the angle $\delta\theta$ around the z axis coming out of the page is

$$\vec{Q}_{\mathrm{rot-z}} = a\delta\theta\,(1, -3, 1, 3, -2, 0)\,. \tag{2}$$

Here we have chosen the long-length of the triangle to be $6a$ and the height of the triangle to be $3a$, the origin is taken to be the center of mass.



(b) Write down the other zero modes parametrized by the coordinates X_{cm} and Y_{cm}.

Vibrational Modes:

(c) Under a reflection over the y axis the displacements $\vec{Q}$ are mapped to some new dispacements $\underline{\vec{Q}}$. Explain qualitatively why $\underline{\vec{Q}}$

$$\vec{Q} \to \underline{\vec{Q}} = (\underline{x}_1, \underline{y}_1, \underline{x}_2, \underline{y}_2, \underline{x}_3, \underline{y}_3) = (\,-x_2, y_2, -x_1, y_1, -x_3, y_3)\,.$$

We say that a vector is *odd* under reflection if $\underline{\vec{Q}} = -\vec{Q}$ and even under reflection if $\underline{\vec{Q}} = \vec{Q}$. Since the problem is symmetric under reflections, the eigenmodes will be either even or odd. The rotation in Eq. (2) is an eigen mode with zero eigenvalue. Is this mode even or odd?

(d) Show that there is only one *odd* basis vector (parameterized by a coordinate $q_{\mathrm{o}}(t)$) which is orthogonal to the three zero modes and determine its form. Then write down two (somewhat arbitrary) *even* basis vectors parameterized by two generalized coordinates $q_1(t)$ and $q_2(t)$ which are orthogonal to the zero modes, which you will use to parametrize the even oscillations.

(e) Write down the Lagrangian of the system using the six well chosen coordinates $(X_{\rm cm}, Y_{\rm cm}, \delta\theta, q_o, q_1, q_2)$ instead of $(x_1, y_1, x_2, y_2, x_3, y_3)$.

(f) Find the eigen-frequencies of the system and qualitatively sketch the non-zero vibrational modes. You should find

$$\omega^2 = \frac{3k}{m}, \frac{2k}{m}, \frac{k}{m} \tag{3}$$

Problem 2. (Landau) Forced oscillations the easier complex way

(a) Determine the retarded green function of the following equations:

(i)

$$\frac{da}{dt} - i\omega_0 a = 0 \tag{4}$$

(ii)

$$\ddot{x} + \eta\dot{x} = 0 \tag{5}$$

(b) Consider the driven harmonic oscillator

$$\ddot{x} + \omega_0^2 x = \frac{f(t)}{m} \tag{6}$$

Write it as an equation for $a = \dot{x} + i\omega x$, and use the green function of (a) to find the specific solution, $a(t)$.

(c) Consider the specific force

$$f(t) = \begin{cases} F_0 & -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases} . \tag{7}$$

Determine and plot the energy in the oscillator for $t \to \infty$ as a function of $\omega_0\tau$. You should find

$$E = \frac{2F_0^2}{m\omega_0^2} \sin^2(\omega_0\tau/2) \tag{8}$$

Give a simple interpretation of the answer for the limit when $\omega_0\tau$ small but finite.

Problem 3. (Laurence Yaffe) A driven set of oscillators

General Background: Consider a set of coupled harmonic oscillators interacting with external time dependent forces. The oscillator Lagrangian without the forces reads[1]

$$L_0 = \sum_{ij} \frac{1}{2} M_{ij} \dot{q}^i \dot{q}^j - \frac{1}{2} K_{ij} q^i q^j \,. \tag{9}$$

The Lagrangian for the forces driving the system is

$$L_{\rm int} = \sum_i F_i(t) q^i \,, \tag{10}$$

and the total Lagrangian is $L = L_0 + L_{\rm int}$. As always, switch coordinates to the eigen basis of the generalized eigenvalue problem

$$q^i = \sum_a E^i_a Q^a \,, \tag{11}$$

where the $\vec{E}_a$ is the a-th eigen-vector of the generalized eigenvalue problem, $K\vec{E}_a = \lambda_a M \vec{E}_a$. Recall that the natural frequency assoicated with the a-th normal mode is $\lambda_a = \omega_a^2$, and the eigenvectors are orthonormal with the mass matrix as weight:

$$\sum_{ij} E^i_a M_{ij} E^j_b = \delta_{ab} \,. \tag{12}$$

(a) Determine the Lagrangian for the coordinates Q^a, and show that the resulting equation of motion is

$$\ddot{Q}^a + \omega_a^2 Q^a = F_a \,, \tag{13}$$

where $F_a = \sum_i F_i E^i_a$ is the projection of the force vector $\vec{F} = (F_i)$ onto the a-th normal mode, i.e. $F_a = \vec{F} \cdot \vec{E}_a$.

Problem: Now consider two masses, $m_1 = 2m$ and $m_2 = m$, are suspended in a uniform gravitational field g by identical massless springs with spring constant k. Assume that only vertical motion occurs, and let z_1 and z_2 denote the vertical displacement of the masses from their equilibrium positions, increasing in the downward direction as shown. An external time-dependent force $F(t)$ is applied to the lower mass (with $F > 0$ indicating a downward vertical force). Assume that the external force vanishes as $t \to \pm\infty$, with the system initially at rest in its equilibrium configuration at time[2] $-\infty$.

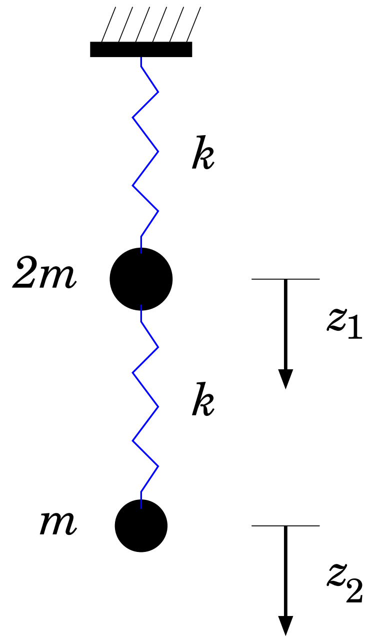
[1]For the rest of this problem we will not use the summation convention.

[2]For example take

$$F(t) = F_0 e^{-|t|/\tau} \tag{14}$$

Let $F(\omega)$ denote the Fourier transform of $F(t)$

$$F(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} F(t) = \frac{2F_0\tau}{1 + (\omega\tau)^2} \tag{15}$$



(b) Construct the Lagrangian for the system, including the external force, and find the resulting equations of motion.

(c) Solve for the motion of both masses (expressed as an integral involving the time-dependent force).

(d) Find the total work done on the system by the external force, $W = E(+\infty) - E(-\infty)$. Show that it can be expressed in the form

$$W = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \chi(\omega) |F(\omega)|^2 \tag{16}$$

with $\chi(\omega)$ real and positive. $\chi(\omega)$ is known as the spectral density, and will be proportional to a sum delta-functions at the resonance frequencies of the system in the absence of damping.

(e) If a small damping term is added to each equation of motion, so $m_i \ddot{z}_i \to m_i \ddot{z}_i + m_i \eta \dot{z}_i$. How are the results of (c) modified.

(f) With the dissipation described in the previous item, again find the total work done by the force on the system. (W is not equal to $E(\infty) - E(-\infty)$, since the work done is ultimately dissipated away.) Show again that the work can be expressed as

$$W = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \chi(\omega) |F(\omega)|^2 \tag{17}$$

Sketch $\chi(\omega)$ with and without damping on the same plot.