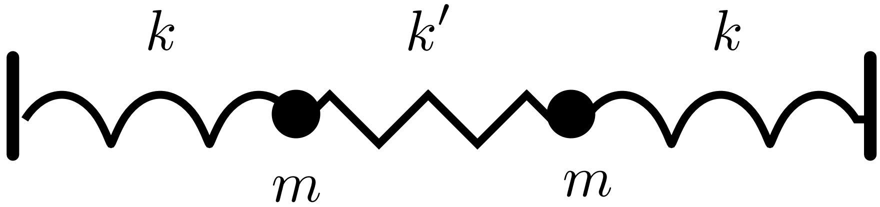
Problem 1. Oscillations with similar frequencies

Consider two particles of mass m coupled to the walls via springs with spring constant $k = m\omega_0^2$. The two particles are weakly coupled by a third spring with spring constant $k' = m\omega'^2$ as shown below. The particles can move only in the x-direction, and the springs are unstretched when the system is at rest. Assume that $\omega' \ll \omega_0$.



(a) (3 points) If at time $t = 0$ the left particle is displaced by an initial position x_0 and the right particle is at rest, determine the subsequent oscillations of the system.

(b) (4 points) Plot qualitatively $x_1(t)$ and $x_2(t)$ in regime where $k' \ll k$. Show all relevant features. Answer qualitatively the following question: given a signal which is a sum of sinusoids

$$A\cos(\omega_1 t) + B\cos(\omega_2 t + \phi) \tag{1}$$

what is required to have prominent beats?

Now consider the case when the particles also experience dissipation. The drag force on the particles is

$$F_{\text{drag}} = -m\eta\frac{dx}{dt}\,, \tag{2}$$

and the drag coefficient is small $\eta \ll \omega' \ll \omega_0$. Starting at t=0, external forces are applied to the particles. The forces on the first and second particles are $F(t)$ and $-F(t)$ respectively. The particles are at rest for $t < 0$.

(c) (6 points) Determine the positions of the particles for $t > 0$ as an explicit integral over $F(t)$.

(d) (3 points) Determine the energy of the system for $t > 0$ as a double integral over $F(t)$.

(e) (4 points) If $F(t)$ is a time-dependent random force satisfying[1]

$$\begin{aligned}\langle F(t)\rangle =& 0\,,\\ \langle F(t)F(t')\rangle =& 2Tm\eta\delta(t-t')\,.\end{aligned}$$

Determine how the energy of the system evolves in time.

Here T is a constant parameter that can be interpreted as the temperature of an external bath provided the force $F(t)$.

[1] Imagine discrizing the system into steps of size Δt, the force in each Δt is $F(t) = \pm 2Tm\eta/\sqrt{\Delta t}$ where each sign occurs with 50% probability.

Problem 2. A non-linear oscillator

An oscillator of mass m and resonant frequency ω_0 has a damping force $F_D = -\beta v^3$ with $\beta > 0$. The motion is initialized with amplitude a_0 and no velocity at time $t = 0$.

(a) Define suitable dimensionless variables so that a dimensionless version of the equation reads:

$$\frac{d^2\overline{x}}{d\overline{t}^2} + \overline{x} + \epsilon \left(\frac{d\bar{x}}{d\bar{t}} \right)^3 = 0 \tag{3}$$

What is the condition on β that the non-linear term may be considered small?

(b) If the oscillator starts at $\bar{t} = 0$ with $\bar{x} = 1$ with $d\bar{x}/d\bar{t} = 0$, use secular perturbation theory to determine approximate behavior of $\bar{x}(\bar{t})$. Show in particular that the amplitude decreases as $\bar{t}^{-1/2}$ at late times. Use Mathematica or other program to determine the exact numerical solution[2], and plot the exact and approximate solution for $\epsilon = 0.3$ up to a time $\bar{t} = 160$.

[2] Look up `NDSolve` and figure it out. I find the following Mathematica advice (parts I and II) by my friend and colleague Mark Alford useful.

Problem 3. Anharmonic oscillations to quadratic order

Consider the oscillator with energy E in the potential

$$U = \frac{1}{2}m\omega_0^2 q^2 + \frac{c}{3}q^3 \tag{4}$$

where the anharmonic contribution is small. The oscillator is at the top of its arc at $t = 0$. We will determine an approximation to $q(t)$

$$q(t) = q^{(0)} + q^{(1)} + q^{(2)} \tag{5}$$

to second order in c.

(a) Choose an appropriate set of units so that the equation of motion can be written with

$$\frac{d^2\bar{q}}{d\bar{t}^2} + \bar{q} + \bar{c}\,\bar{q}^2 = 0\,, \tag{6}$$

with initial condition $\bar{q}(0) = 1$. $\bar{q}$, $\bar{c}$ and $\bar{t}$ are dimensionless versions of q, c and t. To lighten the notation we will drop the bars for the remainder of this problem. $\bar{c}$ is small in this problem; what does this imply for c?

(b) Solve for $q^{(0)}$, $q^{(1)}$, and $q^{(2)}$. You should find to order c^2

$$q(t) = a\cos(\omega t) - \frac{a^2 c}{2} + \frac{a^2 c}{6}\cos(2\omega t) + \frac{a^3 c^2}{48}\cos(3\omega t) \tag{7}$$

with

$$\omega = 1 - \frac{5c^2}{12} + \ldots \tag{8}$$

and amplitude a adjusted to reproduce the initial condition $q(0) = 1$:

$$1 = a - \frac{a^2}{2}c + \frac{a^2 c}{6} + \frac{a^3 c^2}{48} \tag{9}$$

or

$$a(c) = 1 + \frac{1}{3}c + \frac{29}{144}c^2 + \ldots \tag{10}$$

(c) The graph in Fig 1 compares solution in Eq. (7) to a numerical solution. Explain why the perturbative solution fails qualitatively for $c = 0.55$.

(d) The motion is periodic with period T. Qualitatively sketch the power spectrum, i.e. if $q(t)$ is expanded in a Fourier series, $q(t) = \sum_n q_n e^{-i2\pi nt/T}$, sketch $|q_n|^2$ versus n. How does increasing the non-linearity c change this spectrum?

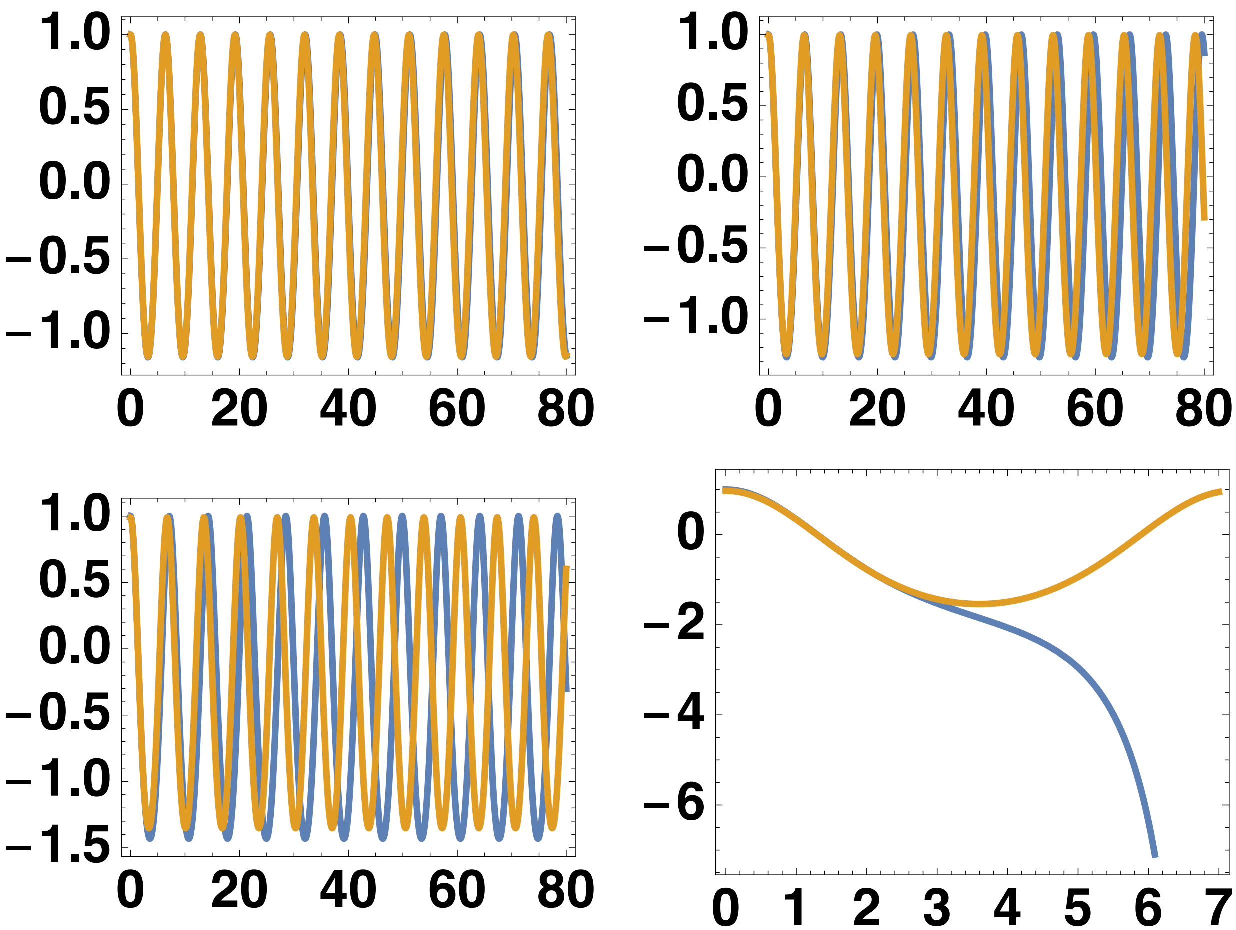


Figure 1: Analytical perturbative solution (yellow curve) compared to the numerical solution (blue curve) versus time for $q(t)$. Reading the graphs like words in a book, the comparison is for $c = 0.2, 0.3, 0.4$ and 0.55 (so $c = 0.3$ is the top right graph).