Problem 1. (Likharev) An effective mass

For a system with the following Lagrangian: Consider an oscillator with generalized coordinate q and resonant frequency ω_0 with an effective mass which is weakly coordinate-dependent $m_{\rm eff} = m(1 + \epsilon q^2)$ where m is a constant[1] . The lagrangian is

$$L = \frac{1}{2} m_{\rm eff}(q)\dot{q}^2 - \frac{1}{2} m\omega_0^2 q^2 \tag{1}$$

Calculate the frequency of oscillations using secular perturbation theory, and from an integral given in class for the period of one dimensional systems (see "Motion of 1d systems" online). Assume that the amplitude of the oscillations is A and that $\epsilon A^2 \ll 1$. You should find by both methods that the predominant frequency of oscillation is

$$\omega \simeq \omega_0 \left(1 - \frac{A^2\epsilon}{4}\right) . \tag{2}$$

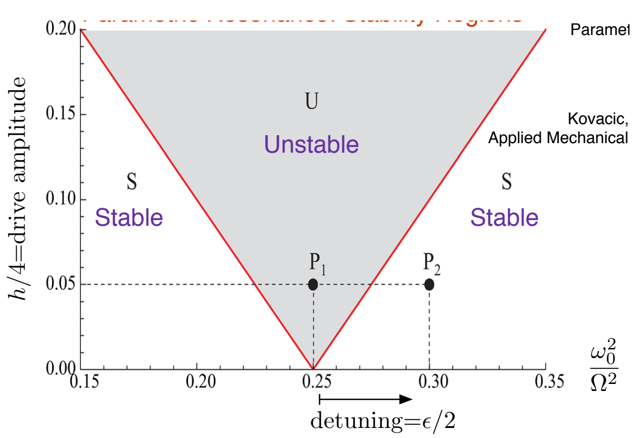
[1] Technically m has units, $[m] = \mathrm{kg} \cdot \mathrm{m}^2$ and q is dimensionless

Problem 2. Parametric resonance with damping

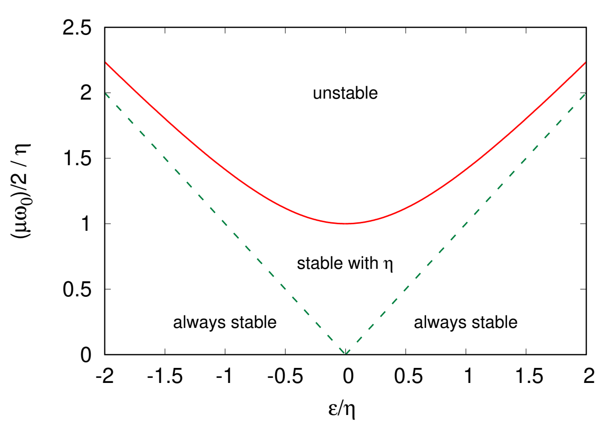
Consider an oscillator with a small damping coefficient η, and a time dependent mass, $m(t) = m_0(1 + \mu\cos(\Omega t))$, with μ is small. The frequency is $\Omega \simeq 2\omega_0 + \epsilon$ with ϵ also small. Thus the equation of motion is

$$\frac{d}{dt}(m(t)\dot{q}) + m_0\omega_0^2 q(t) + m_0\eta\dot{q} = 0 \tag{3}$$

Determine the regions in the ϵ, μ plane where the oscillations are stable and unstable. How is the plot from class (the first plot below) modified by the non-zero damping coefficient?



You should find the following picture:



Problem 3. A pendulum in a harmonic electric field

A simple pendulum consists of a particle of mass m at the end of weightless rod of length ℓ. The particle has a charge q and sits in an electric field of amplitude E_0, directed in the horizontal dirction, which oscillates rapidly with frequency Ω, $\Omega \gg \sqrt{g/\ell}$.

(a) Determine the Lagrangian for this system, and the equation of motion.

(b) Above a critical field strength E_c the position $\phi = 0$ (the bottom of the pendulum) becomes unstable. Determine E_c and determine the new point of stability for $E > E_c$. Sketch the effective potential for $E < E_c$ and $E > E_c$.

(c) Analyze the validity of your approximations. Is the critical field large or small compared to mg/q when your approximation is valid?