Problem 1. (Optional) A tutorial on Fourier Transforms

This is intended as a warm up for a number of problems on waves. Answer all parts *as briefly as possible*. All graduate students must know this *all* of this stuff if they want to succeed.

Define the fourier transform the physicist way:

$$F(k) = \int_x e^{-ikx} f(x) \qquad f(x) = \int_k e^{ikx} F(k) \tag{1}$$

Here integrals over wavenumbers k mean the following

$$\int_x = \int_{-\infty}^{\infty} dx \qquad \int_k \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tag{2}$$

This own little notation that you might try – it is covenient since $\int_k \int_x e^{ikx} = 1$. Recall that the integral of a pure phase is a delta function:

$$\int_x e^{-ikx} = 2\pi\delta(k) \qquad \int_k e^{ikx} = \delta(x) \tag{3}$$

We use

$$f(x) \leftrightarrow F(k) \tag{4}$$
$$g(x) \leftrightarrow G(k) \tag{5}$$

to indicate Fourier transform pairs. a is a postive constant of unit length. b is a positive constant of unit wavenumber k. ϵ is a small real constant

(a) (**Examples:**) Consider the following most useful Fourier transforms that every serious physicist (experimentalist and theorist) knows by memory:

$$\frac{1}{2a}\exp(-|x|/a) \quad \leftrightarrow \quad \frac{1}{1+(ka)^2} \tag{6}$$
$$\frac{1}{a}\text{step}(x/a) \quad \leftrightarrow \quad \text{sinc}(ka) \equiv \frac{\sin(ka/2)}{(ka/2)} \tag{7}$$
$$\frac{1}{\sqrt{2\pi a^2}}\exp(-x^2/(2a^2)) \quad \leftrightarrow \quad \exp(-\frac{1}{2}a^2k^2) \tag{8}$$
$$\theta(x)e^{-bx} \quad \leftrightarrow \quad \frac{1}{b+ik} \tag{9}$$

Here $\text{step}(x) = \theta(x+1/2) - \theta(x-1/2)$ (see the first panel of Fig. 1) is the square wave function with integral one. Try to think of a way to remember these. For instance the second one is what comes out of a single slit diffraction of experiment. If you don't know how to do these integrals try to fix that. The table can be read either way, with the replacements $k \to -x$ and an additional factor of 2π, e.g.

$$\frac{1}{2\pi}\frac{1}{1+(xb)^2} \quad \leftrightarrow \quad \frac{1}{2b}e^{-|k|/b} \tag{10}$$

In this case b has units 1/length and is the width in k-space.

Prove that the first row of this table holds. Graph $\exp(-|x|/a)$ and its Fourier transform for several values of ϵ, with $\epsilon \equiv 1/a$, and $a \to \infty$ or $\epsilon \to 0$. Argue that this Fourier transform is 2π times a Dirac sequence[1]:

$$2\pi \left[\frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + k^2} \right] = 2\pi \delta_\epsilon(k) \tag{11}$$

Whenever you see a δ-function it must be remembered that it is shorthand for a Dirac sequence $\delta_\epsilon(x)$. The identity

$$\int_x e^{-ikx} = 2\pi\delta(k) \tag{12}$$

is shorthand for a limiting process where the Fourier integral is cutoff in some way. Here we have explored the cutting it off like this

$$\lim_{\epsilon \to 0} \int_x e^{ikx} e^{-\epsilon|x|} = 2\pi\delta_\epsilon(k) \tag{13}$$

but it can be cutoff in many ways.

(b) (**Real/Imaginary/Even/Odd**) Consider the following table:

If ...	then ...
$f(x)$ is real	$F(-k) = (F(k))^*$
$f(x)$ is imaginary	$F(-k) - F(k)$
$f(x)$ is even ($f(-x) = f(x)$)	$F(-k) = F(k)$ (i.e. F is even)
$f(x)$ is odd ($f(-x) = -f(x)$)	$F(-k) = -F(k)$ (i.e. F is odd
$f(x)$ is real and even	$F(k)$ is real and even
$f(x)$ is real and odd	$F(k)$ is imaginary and odd
$f(x)$ is imaginary and even	$F(k)$ is imaginary and even
$f(x)$ is imaginary and odd	$F(k)$ is real and odd

Table 1:

Prove the first line and state what this means for the even and odd properties of the real and imaginary parts of $F(k)$. Prove the sixth line as well.

(c) (**Shifting**) Consider the Fourier transform pair $f(k) \leftrightarrow F(k)$. We have the following properties

$$f(x)e^{ik_0x} \quad \leftrightarrow \quad F(k-k_0) \quad \text{wavenumber shifting} \tag{14}$$

$$f(x-x_0) \quad \leftrightarrow \quad F(k)e^{-ikx_0} \quad \text{spatial shifting} \tag{15}$$

Prove the first of these.

[1]A Dirac sequence is any family of functions $\delta_\epsilon(x)$, labelled by a parameter $\epsilon \to 0$, which satisfies $\int_{-\infty}^{\infty} \delta_\epsilon(x) = 1$, and approaches zero for $|x| \neq 0$ and fixed.

(d) (**Scaling**) Consider the Fourier transform pair $f(k) \leftrightarrow F(k)$.

$$f(xa) \quad \leftrightarrow \quad \frac{1}{|a|}F(k/a) \qquad \text{spatial scaling} \tag{16}$$

$$\frac{1}{|b|}f(x/b) \quad \leftrightarrow \quad F(bk) \qquad \text{wavenumber scaling} \tag{17}$$

Prove the first of these and qualitatively describe the relation to the uncertainty principle.

(e) (**Derivatives**) Show that

$$\int_x f(x) = F(k)\Big|_{k=0}, \tag{18}$$

and show more generally that the moments of $f(x)$ are related to to the taylor series of $F(k)$ at the origin by

$$\int_x f(x)(-ix)^n = \left(\frac{d}{dk}\right)^n F(k)\Big|_{k=0} \tag{19}$$

The converse also holds the moments of $F(k)$ are related to the derivatives of $f(x)$ at the origin

$$\int_k F(k)(ik)^n = \left(\frac{d}{dx}\right)^n f(x)\Big|_{x=0} \tag{20}$$

(f) (**Analyticity/Asymptotic form**) It follows from Eq. (20), that if a function's Fourier transform $F(k)$ falls slower than $1/k^n$, then its n-th derivative will not generically exist. Generally analytic functions (which have all derivatives) will have a Fourier transform which decreases exponentially at large k. Consider the Fourier transform of

$$\frac{1}{1+(ka)^2} \tag{21}$$

How does the result corroborate this theorem?

(g) (**Convolutions**) Consider the convolution of two functions

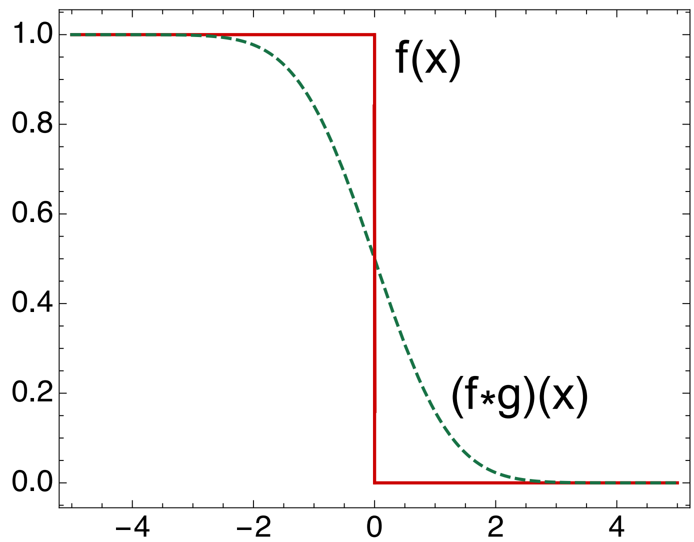
$$(g * f)(x) \equiv \int_{-\infty}^{\infty} dy\, g(x-y)\, f(y) \tag{22}$$

Usually the convolution is used to provide a transformation or *response* to the source function $f(y)$. For instance if $g(z)$ is a normalized narrow gaussian:

$$g(z) = \frac{1}{\sqrt{2\pi a^2}} \exp(\frac{-z^2}{2a^2}) \tag{23}$$

then convolution process, just replaces any function $f(x)$ with a kind of average of all of its neighboring values. The figure below shows a step-like function $f(x)$ convolved with a gaussian, $g(z)$.

Most often in physics $f(y)$ is some kind of source and $g(x-y)$ is the Green function, *i.e.* the value at x due to a delta-fcn source at position y. For instance if $f(y)$ is the charge density $\rho(y)$ then convolution of $\rho(y)$ the green function $1/4\pi|x-y|$ gives the electic field at point x.



(i) Working only in coordinate space show that if $\int_y g(y) = 1$. The integral of f is unchanged by the convolution process.

(ii) Show that the Fourier transform of the convolution is a product of Fourier transforms.

$$(g * f)(x) \leftrightarrow G(k)F(k) \tag{24}$$

(iii) By working in fourier space, show using the convolution theorem and Eq. (18) that if $\int_y g(y) = 1$, then the integral of f is unchanged by the convolution process.

(iv) Compute the fourier transform of

$$\left(\frac{\sin(ka/2)}{(ka/2)}\right)^2 \tag{25}$$

by using the convolution theorem. You can check your result from the its integral in coordinate space.

Describe qualitatively (using the convolution theorem) what are the functions $B_n(x)$ which are defined by Fourier transform of $(\mathrm{sinc}(ka))^n$

$$B_n(x) \leftrightarrow \left(\frac{\sin(ka/2)}{ka/2}\right)^n \tag{26}$$

These functions $B_n(x)$ are known as B splines and are important for numerical work. Note: the higher the n, the faster it falls in k-space, the smoother the function. Fig. 1 shows the first couple B splines.

(v) Consider the convolution of a smooth function $f(x)$ with a normalized gaussian of width a which is small compared to the scales of $f(x)$. By working in k-space, show that

$$(f * g)(x) \simeq f(x) + f''(x)\frac{a^2}{2} \tag{27}$$

Qualitatively, what does $f''(x)$ the term do?

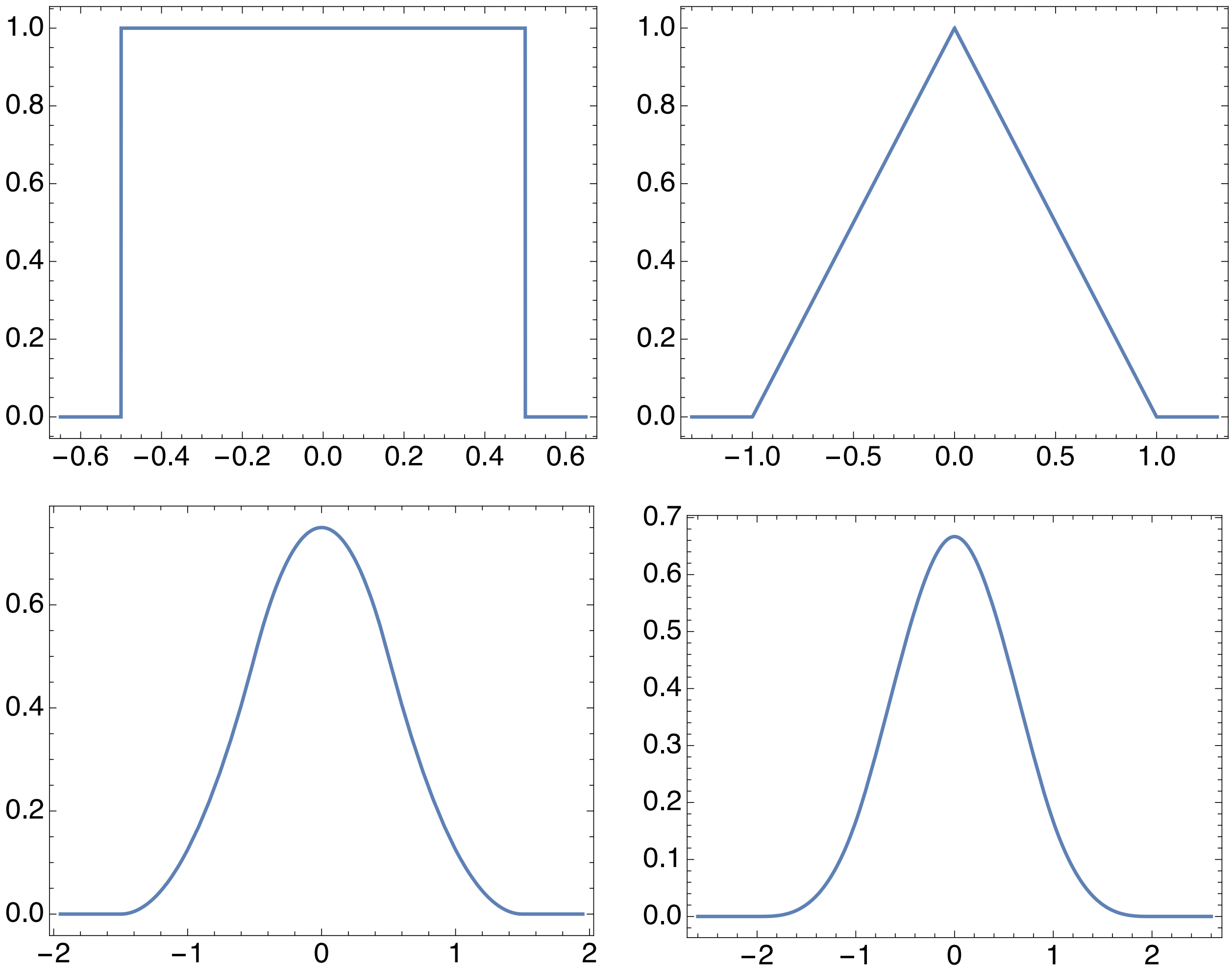


Figure 1: The first four B-splines, $B_1 \ldots B_4$, laid out like a two lines of a book. The second one is continuous but has discontinuous derivatives. The fourth one is the the cubic bspline, which has continuous second derivatives but discontinuous third derivatives.

(h) (**Correlations**) Closely related to the convolution of two functions (but usually somewhat distinct in physical situations) is the *correlation* of two functions:

$$\mathrm{Corr}(f, g)(x) \equiv \int dy\, f(x+y)\, g^*(y)\,, \tag{28}$$

which is is relevant when we want to quantify over what range of lengths, x, a physical observable f is influenced by value g. Often g is a real function and the star is unnecessary. Show that the correlation function satisfies

$$\mathrm{Corr}(f, g) \leftrightarrow F(k)(G(k))^* \qquad \text{Correlation-Theorem} \tag{29}$$

and thus the fourier transform of *auto-correlation function* is the power spectrum $|F(k)|^2$

$$\mathrm{Corr}(f, f) \leftrightarrow |F(k)|^2 \qquad \text{Wiener-Khinchin Theorem} \tag{30}$$

For this exercise you will need to recognize that the Fourier transform of $g^*(x)$ (what I sometimes call $G_*(k)$) is not quite $(G(k))^*$.

(i) (**Parseval**) Finally prove Parsevals theorem

$$\int dx |f(x)|^2 = \int \frac{dk}{2\pi} |F(k)|^2 \qquad \text{Parseval's Theorem} \tag{31}$$

which says that the power can be computed either in coordinate or momentum space.